Stochastic Methods
(exercises)

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1. Find the mathematical expectation and the variance of the random variable (r.v.) η^2, where η is a continuous uniformly distributed r.v. (c.u.d.r.v.) in $[0, 2]$.

   5 points

2. Construct an algorithm for generating a discrete r.v. ξ:

$$\xi = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ p_1 & p_2 & p_3 & p_4 \end{pmatrix},$$

   where $p_i = P\{\xi = x_i\} = \frac{1}{4}$ using a random number γ (γ is a c.u.d.r.v. in $[0, 1]$).

   5 points

3. Construct a plain Monte Carlo algorithm for computing

$$I = \int_0^1 \sqrt[5]{x}dx$$

   using a r.v. θ with a constant density function $p(x) = const$. Show that the variance $D\theta$ is much smaller than I^2.

   10 points

4. Find the mathematical expectation and the variance of the random variable (r.v.) γ^3, where γ is a continuous uniformly distributed r.v. (c.u.d.r.v.) in $[0, 1]$.

5 points

5. Construct an algorithm for generating a discrete r.v. ξ:

$$\xi = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix},$$

where $p_i = P\{\xi = x_i\}$ using a random number γ (γ is a c.u.d.r.v. in $[0, 1]$).

5 points

6. Construct a *Key sampling Monte Carlo algorithm* for

$$I = \int_0^1 e^x dx.$$

Show that the variance of the algorithm $D\theta_0$ is zero.

10 points

7. For calculating the integral

$$I = \int_0^1 e^x dx$$

apply Monte Carlo with a r.v. $\theta' = f_1(x)$ using *symmetrization of the integrand*: $f_1(x) = \frac{1}{2}[f(x) + f(a + b - x)]$. Show that $D\theta' << I^2$.

15 points

8. Consider the integral

$$I = \int_0^1 f(x)p(x)dx,$$

where $f(x) = e^x$ and $p(x) = 1, \quad x \in [0, 1]$.

a) Find the value of I.

b) Construct a Monte Carlo algorithm using *separation of principle part* for $h(x) = x$ and show that $E\theta' = I$.

c) Show that $D\theta' << I^2$.

20 points

9. Consider the integral

$$I = \int\int_{\Omega} (2 - y)dxdy,$$

where $\Omega \equiv \{(x, y) : 0 \leq x \leq 1; \quad 0 \leq y \leq 1 + \frac{x}{4}\}$. Construct an *integration on subdomain Monte Carlo algorithm* assuming that

$$I' = \int\int_{\Omega'} (2 - y)dxdy \quad \text{and} \quad c = \int\int_{\Omega'} p(x, y)dxdy,$$

where $\Omega' \equiv \{(x, y) : 0 \leq x \leq 1; \quad 0 \leq y \leq 1\}$ using a r.v. θ' with a constant density function $p(x, y)$ (such that $\int\int_{\Omega} p(x, y)dxdy = 1$).

Show that:

a) $D\theta' << I^2$;

b) $D\theta' \leq (1 - c)D\theta$, where θ is the corresponding r.v. for the plain Monte Carlo integration.

25 points