Improved lower bounds for embedding hypercubes on de Bruijn graphs

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1 Introduction

In a paper published in JDPC in 1994, Heydemann, Opatrny and Sotteau studied the problem of embedding hypercubes and grids in the de Bruijn graph [3]. Hypercubes and grids are well-known graphs, often used for interconnection networks of parallel machines. The de Bruijn graph has many interesting structural properties including a fixed degree for a logarithmic diameter. it is a very good candidate for the construction of interconnection networks, especially for embedded systems [2]. In this note, we will restrict the result to $B(2,n)$, whose vertexes are defined by all the words of length n over the binary alphabet 01. More details can be found in [4]. One of the most useful result of [2] was to present a constructive method for embedding an hypercube of dimension n (denoted by $H(n)$ into the de Bruijn graph $B(2,n)$ with dilation at most $\lceil \frac{n}{2} \rceil$). The *dilation* is one of the most important parameter for measuring the quality of an embedding. It is defined as the maximum over all the lengths of the paths linking the images of all the pairs of vertexes. A formal definition can be found in [3] or [4].

The main result of this note is to prove that the minimal dilation for embedding $H(n)$ in $B(2,n)$ is between $\lceil \frac{n}{2} \rceil$ and the greatest divisor of n which is less than $\lceil \frac{n}{2} \rceil$. before presenting in details the technical result, let us recall first some basic results.

2 Preliminary results

Proposition 1. [3] If a hypercube $H(n)$ can be embedded into the de Bruijn graph $B(2,n)$ with dilation sp (where s and p are integers and s is a divisor of n) then there exists an embedding of $H(n)$ into $B(2^s, \frac{n}{s})$ with dilation p.

The proof is constructive using an adequate codage of the nodes [3].

An immediate corollary is obtained for the case $p = 1$: $H(n)$ is a sub-graph of $B(2^s, \frac{n}{s})$. This result can be directly expended for minimal dilation. Moreover, it is interesting to notice that this result holds also for embedding grids instead of hypercubes.

Proposition 2. [3] Every embedding of $H(n)$ into $B(2,n)$ has dilation at least:

$$\begin{array}{ll} \lceil \frac{n}{2} \rceil & \text{for} \quad 2 \leq n \leq 6 \\ 3 & \text{for} \quad n \geq 7 \\ 3+\log_3(n-3)/12 & \text{for} \quad n \geq 16 \end{array}$$

Proposition 3. [1] $q = q(n,m)$ determines the smallest integer for which $H(n)$ is a subgraph of $B(q,m)$ (for $m, \quad n \geq 2$). Thus:

- $q(n,@) = 2^i$ if $n = 2i$ for some integer $i \geq 1$;
- $q(n,2) = 3.2^{i-1}$ if $n = 2i+1$ fore some integer $i \geq 1$;
- $q(n,m) = 2^{n-1}$ if $m \geq 3$.

3 Improved lower bound

Let k be the minimal dilation for embedding hypercube $H(n)$ into $B(2,n)$. We denote by n_1 the greatest divisor of n which is less than $\lceil \frac{n}{2} \rceil$. We denote by n_1 the greatest divisor of n which is less than $\lceil \frac{n}{2} \rceil$. If n is odd n_1 is the greatest divisor of n which is different of n. If n is even n_1 is the greatest divisor of n for which $n_1 < \frac{n}{2}$.

First, notice that from [3], it is straightforward to observe that $k \leq \lceil \frac{n}{2} \rceil$. We state now the main result of this note.

Lemma. k is strictly greater than n_1.

Proof

Case 1. let k be a divisor of n, then the dilation for embedding $H(n)$ into $B(2^k, n/k)$ is 1 (from Proposition 1), i.e. $H(n)$ is a subgraph of $B(2^k, n/k)$. let $k < \lceil n/2 \rceil$, then $n/k \geq 3$. From Proposition 3 the smallest number for which $H(n)$ is a subgraph of $B(q, n/k)$ is $q = 2^{n-1}$. But $k < \lceil n/2 \rceil$ therefore $2^{n-1} > 2^k$. It means that $H(n)$ can not be a subgraph of $B(2^k, n/k)$ and this contradiction comes from the hypothesis that $k < \lceil n/2 \rceil$. Thus we prove that k can not be less then $\lceil n/2 \rceil$, it means that $k = n/2$ if n is even or does not exists if n is odd.

Case 2. Let k is not a divisor of n. Suppose there exists an integer k_1 with $k < k_1 < \lceil n/2 \rceil$ and k_1 is a divisor of n. From Proposition 1, the dilation for embedding $H(n)$ into $B(2^{k_1}, n/k_1)$ is 1, i.e. $H(n)$ is a subgraph of $B(2^{k_1}, n/k_1)$. But $k_1 < \lceil n/2 \rceil$ then $n/k_1 \geq 3$ and from Proposition 3 the smallest number for which $H(n)$ is a subgraph of $B(q, n/k_1)$ is $q = 2^{n-1}$. But $2^{n-1} > 2^{k_1}$, it means that $H(n)$ cannot be a subgraph of $B(2^{k_1}, n/k_1)$ and this contradiction comes from the hypothesis that there exists a divisor of n, which is greater than k and less than $\lceil n/2 \rceil$.

By Case 1 and Case 2 we prove that between k and $\lceil n/2 \rceil$ there are not a divisor of n and if k is a divisor of n then $k = n/2$ if n is even or does not exist. Therefore k is strictly greater than n_1.

We give now a characterization of n_1.

Proposition 4.

Let $n = p_1^{\alpha_1} \dots p_l^{\alpha_l}$, where $p_1 < p_2 < \dots < p_l$ are prime numbers. The greatest divisor of n which is less than $\lceil \frac{n}{2} \rceil$ is:

- $n_1 = \frac{n}{p_1}$ if $p_1 > 2$
- $n_1 = \frac{n}{p_1^2}$ if $p_1 = 2, \quad \alpha_1 > 1$ and $p_2 > 3$
- $n_1 = \frac{n}{p_2}$ if $p_2 = 3$ or $\alpha_1 = 1$

4 Summary and Conclusion

we give below some specific values in a synthetic table for giving a better insight for comparing the results presented in this note with existing results:

n	[3]	New lower bound
12	3	5
15	3	6
21	4	8
$3^2 4 + 3$	24	$\frac{3^2 4+3}{3} + 1$

For most values of n, we improved considerably the existing lower bounds for embedding $H(n)$ into $B(2, n)$.

References

[1] T. Andreae and M. Hintz, On hypercubes in de Bruijn graphs. *Communication at In. Workshop on Interconnection Networks, IWIN'97*, Praha, July 1-5, 1997.

[2] J. C. Bermond and C. Peyrat, de Bruijn and Kautz networks: a competitor for the hypercube?*Proceedings of first european workshop on hypercube and distributed computers*, Rennes, 279-294, 1989.

[3] M. C. Heydemann, J. Opatrny and D. Sotteau, Embeddings of hypercubes and grids into de Bruijn graphs. *JPDC*, 23, 104-111,1994.

[4] F. Thomson Leighton, Introduction to parallel algorithms and architectures. Morgan Kaufmann publishing, 1994.