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IICT – BAS eISSN: 2367-8666

Lecture Notes in Computer Sciences and Technologies

The Best Uniform Rational Approximation: Application to Solving Equations Involving Fractional Powers of Elliptic Operators

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eISBN: 978-619-7320-08-4

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eISSN: 2367-8666

http://parallel.bas.bg/lcst/

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Preface

The handbook is intended to be used as a complementary part of the courses on “Numerical Methods for Sparse Linear Systems” and “Convex Analysis and its Applications to Image Processing” that are currently taught within the Master’s program “Computational Mathematics and Mathematical Modelling” in Sofia University “St. Kliment Ohridski”. The presented material can also serve as a short course on numerical methods for solving fractional diffusion problems and as an extension of existing master or PhD programs in the field of advanced scientific computations. The handbook considers the mathematical problem of solving systems equations involving fractional powers of self-adjoint elliptic operators. Due to the mathematical modelling of various non-local phenomena using such operators, a number of numerical solution methods have been introduced, studied, and tested in the last two decades. The handbook deals with the discrete counterpart of such problems obtained from finite difference or finite element approximations of the corresponding elliptic problems. All necessary information regarding the recently introduced by the authors’ method based on best uniform rational approximation (BURA) of a proper scalar function in the unit interval is provided. A substantial part of the handbook is the presented 160 tables containing the related coefficients, zeros, poles, extreme points of the error function, the terms of the BURA partial fraction decomposition, etc. Links to website where one can download the files with the data characterizing the particular rational approximations, which are available with enough significant digits, are also provided so one can use them in his/her own computations. The examples are presented in a form ready for computer exercises.

Chapter 1

Theoretical background

1.1 Introduction to equations with nonlocal operators

1.1.1 Overview of problems and mathematical models involving nonlocal operators

The general area of non-local operators in mathematics has long and interesting history. However, its recent progress is due to the remarkable and sometimes surprising applications in physics, engineering, mechanics, biology and finance.

Time dependent models Transient diffusion is one of the most prominent transport of energy mechanisms found in nature. The classical diffusion model is based on the assumption that the mean squared particle displacement grows linearly with in time. This phenomenon is called Brownian motion. However, a number of experimental studies in engineering, physics, biology and finance indicate that the Brownian motion assumption may not be adequate for accurately describing some physical processes, and the mean squared displacement can grow either sub-linearly or super-linearly with the time. These phenomena are known in the literature as sub-diffusion and super-diffusion, respectively. These experimental studies include electron transport in Xerox photocopier [53], visco-elastic materials [12, 24], thermal diffusion in fractal domains [45], column experiments [30], protein transport in cell membrane [38], etc. The underlying stochastic process for sub-diffusion and super-diffusion is usually given by continuous time random walk and Lévy process, respectively, and the corresponding macroscopic model for the probability density function of the particle appearing at certain time instance and its space location is given by a diffusion model with a fractional-order derivative in time and in space, respectively. These is used in modeling dynamics in fractal media, [57], or modeling materials with memory (e.g. viscoelasticity, Bagley-Torvik equation, [59]). For more details, interesting discussion, and an extensive list of practical applications and physical modeling in engineering, physic, and biology we refer to the excellent surveys [42, 43].

Nonlocal models involving non-locality in either time or space have evolved into an active and fruitful research area. The great success is due to the joint efforts of researchers in mathematical analysis, partial differential equation theory, numerical analysis, stochastic analysis, and other applied disciplines. The sub-diffusion is just one example of nonlocal problems. We refer interested readers to the excellent surveys for other nonlocal problems and applications, namely, on problem involving fractional (spectral and integral) Laplacian [39, 4], on inverse problems with anomalous diffusion [34, 51], on regularity theory of nonlocal elliptic equations in bounded domains [49], and on nonlocal problems arising in peridynamics [17]. For an early overview on the numerical methods for fractional-order ordinary differential equations, we refer to the paper [14].

More recently the technique involving non-local operators has been used in mathematical modeling of optimal control and inverse problems, stochastic processes, and image reconstruction. The literature is vast and the research is growing fast, especially in the area of scientific and engineering applications. Our work is part of the progress in the numerical analysis of such problems, summarized in the recent special issue of *Journal of Computational Physics* [35].

Motivated by the phenomenal success of this technique in mathematical modeling of various applied problems, there has been an explosive growth in the numerical algorithms and mathematical analysis of various sub-diffusion models. This includes high order approximations, spectral methods, fast solution techniques, etc and their practical justification and theoretical analysis. An interesting phenomenon of sub-diffusion equations is the different smoothing properties of the solution operator and lack of regularity of the corresponding solution. This has been address in the recent paper by M. Stynes, [56] with serious implication to the construction and the analysis of the numerical method for such problems. For the progress in this area we direct the interested reader to the special issue on the topic at *Computational Methods in Applied Mathematics* [32] and for the concise review [33] for the recent progress in the area of numerical methods for fractional evolution equations.

Steady-state models "Fractional" order differential operators appear naturally in many areas in mathematics and physics, e.g. trace theory of functions in Sobolev classes (Sobolev embedding, elliptic type), the theory of special classes analytic functions (Dzhrbashyan, [15], Riemann-Liouville fractional derivative, [37, 48]), modeling various phenomena, e.g. particle movement in heterogeneous media, [44] and/or heavily tailed Levy flights of particles, [30], peridynamics (deformable media with fractures), [22], image reconstruction, [23], etc. The most important property of these operators is that they are non-local.

There are two main definitions of "fractional Laplacian" (and more general steady-state sub-diffusion problem) that are used modeling of various non-local diffusion-like problems. For a reveling and thorough discussion on this topic and the relevant numerical methods we refer to [6, 40]. Here we shall consider the case of so-called spectral fractional Laplacian, see Subsection 1.2.1.

The aim of this handbook In this handbook we consider numerical methods and algorithms for solving equations involving fractional powers of self-adjoined elliptic operators, defined either by Dunford-Taylor-like integrals or by the representation through the eigenvalues and the eigenfunctions of the elliptic operator. The numerical methods are done in two basic steps:
(1) approximation of the corresponding elliptic operator by finite elements in a finite dimension space V_h (of dimension N) or similar approximation by finite differences on a rectangular mesh leading to a matrix acting in the Euclidean space $\mathbb{R}^N$; this results in a semi-discrete scheme;
(2) approximation of the fractional powers of the discretized elliptic operator using the best uniform rational approximation (BURA) of a certain function on $[0, 1]$, which results in a fully discrete scheme.

1.1.2 Semi-discrete and fully discrete approximations in the elliptic case

First, in Subsection 1.2.1 we consider systems of equations generated by fractional powers of elliptic operators of the type $\mathcal{A}^\alpha u = f$. Examples of such operators are given in Subsection 1.2.1. Then the fractional powers of $\mathcal{A}$ are defined by Dunford-Taylor integrals, which can be transformed when $\alpha \in (0, 1)$ to the Balakrishnan integral (1.5). Further we also consider other problems like $\mathcal{A}^\alpha u + bu = f$ and initial value problem $\frac{\partial u(t)}{\partial t} + \mathcal{A}^\alpha u(t) = f(t)$, $t > 0$, with $u(0) = v$.

The approximation of such problems is done in two steps. At the first step we discretize the elliptic operator $\mathcal{A}$ by using finite elements (over a finite dimensional space V_h, $dim V_h = N$) or finite differences over a uniform mesh with N points. In the case of finite element approximation we get a symmetric and positive definite operator $\mathbb{A} : V_h \to V_h$, that results in an operator equation $\mathbb{A}^\alpha u_h = f_h$ for $f_h \in V_h$ given and $u_h \in V_h$ unknown. In the case of finite difference approximation we get a symmetric and positive definite matrix $\widetilde{\mathbb{A}} \in \mathbb{R}^{N\times N}$ and a vector $\widetilde{f}_h \in \mathbb{R}^N$, so that the approximate solution $\widetilde{u}_h \in \mathbb{R}^N$ satisfies $\widetilde{\mathbb{A}}^\alpha \widetilde{u}_h = \widetilde{f}_h$. The fractional powers of the operator $\mathbb{A}^\alpha$ and the matrix $\widetilde{\mathbb{A}}^\alpha$ are defined by (1.5) or (1.6) (with finite summation), correspondingly. These equations generate the so-called semi-discrete solutions

$$u_h = \mathbb{A}^{-\alpha} f_h \text{ or/and } \widetilde{u}_h = \widetilde{\mathbb{A}}^{-\alpha} \widetilde{f}_h.$$

In the second step we essentially approximate the Balakrishnan integral (1.5). This is done by introducing the rational function $r_{\alpha,k}(t)$, which is the best uniform rational approximation (BURA) of t^α on $[0, 1]$ (see Subsection 1.5.3) and apply it to produce the fully discrete approximations, see (1.39),

$$w_h = \lambda_1^{-\alpha} r_{\alpha,k}(\lambda_1 \mathbb{A}^{-1}) f_h \quad \text{and} \quad \widetilde{w}_h = \lambda_1^{-\alpha} r_{\alpha,k}(\lambda_1 \widetilde{\mathbb{A}}^{-1}) \widetilde{f}_h.$$

This handbook is devoted to a full characterization of the rational functions $r_{\alpha,k}(t)$, $0 < t < 1$, by providing its poles and partial fraction representation and the extremal points of the error.

Semi-discrete and fully discrete approximations in the parabolic sub-diffusion case Further, in Sections 1.2.2 and 1.2.3 we apply the same strategy to the sub-diffusion-reaction problem (1.7) and time-stepping procedure for the transient sub-diffusion problem (1.8). Both cases result in solving the following type semi-discrete problem (1.21):

$$(\mathbb{A}^\alpha + b\mathbb{I})u_h = f_h \quad \text{or} \quad u_h = (\mathbb{A}^\alpha + b\mathbb{I})^{-1} f_h.$$

In this case, we consider two possible fully discrete schemes. The first one is based on the BURA of the function $t^\alpha/(1 + qt^\alpha)$ on the interval $[\delta, 1]$ with q a non-negative constant. The second fully discrete scheme is based on a rational approximation of the same function (NOT the best one), called further URA-method. For details, see Subsection 1.5.6 and Definition 1.5.2.

1.2 Examples of problems with fractional powers of elliptic operators

1.2.1 Spectral fractional powers of elliptic operators

In this handbook we consider the following second order elliptic equation with homogeneous Dirichlet data:

$$\begin{aligned} -\nabla \cdot (a(x)\nabla v(x)) + b(x)u = f(x), \qquad & \text{for } x \in \Omega, \\ v(x) = 0, \qquad & \text{for } x \in \partial\Omega. \end{aligned} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 1$. We assume that $0 < a_0 \leq a(x)$ and $b(x) \geq 0$ for $x \in \Omega$.

The fractional powers of the elliptic operator associated with the problem (1.1) are defined in terms of the weak form of (1.1), namely, $v(x)$ is the unique function in $V = H_0^1(\Omega)$ satisfying

$$a(v, \theta) = (f, \theta) \qquad \text{for all } \theta \in V. \tag{1.2}$$

Here

$$a(w, \theta) := \int_\Omega (a(x)\nabla w(x) \cdot \nabla\theta(x) + bw\theta)\, dx \quad \text{and} \quad (w, \theta) := \int_\Omega w(x)\theta(x)\, dx.$$

For $f \in L^2(\Omega) := X$, (1.2) defines a solution operator $Tf := v$. Following [36], we define an unbounded operator $\mathcal{A}$ on X as follows. The operator $\mathcal{A}$ with domain

$$D(\mathcal{A}) = \{Tf \, : \, f \in X\}$$

is defined by

$$\mathcal{A}v = g \quad \forall v \in D(\mathcal{A}), \quad \text{where} \quad g \in X \quad \text{with} \quad Tg = v. \tag{1.3}$$

The operator $\mathbb{A}$ is well defined as T is injective.

Thus, the focus of our work in this handbook is numerical approximation and algorithm development for the equation:

$$\mathcal{A}^\alpha u = f \quad \text{with a solution} \quad u = \mathcal{A}^{-\alpha} f. \tag{1.4}$$

Here $\mathcal{A}^{-\alpha} = T^\alpha$ for $\alpha > 0$ is defined by Dunford-Taylor integrals which can be transformed when $\alpha \in (0, 1)$, to the Balakrishnan integral, e.g. [3]: for $f \in X$,

$$u = \mathcal{A}^{-\alpha} f = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \mu^{-\alpha}(\mu\mathcal{I} + \mathcal{A})^{-1} f \, d\mu. \tag{1.5}$$

This definition is sometimes referred to as the spectral definition of fractional powers. One can also use an equivalent definition through the expansion with respect to the eigenfunctions ψ_j and the eigenvalues λ_j of $\mathcal{A}$, e.g. [2, 40]:

$$\mathcal{A}^\alpha u = \sum_{j=1}^\infty \lambda_j^\alpha (u, \psi_j)\psi_j \;\; \text{so that } u = \sum_{j=1}^\infty \lambda_j^{-\alpha} (f, \psi_j)\psi_j \tag{1.6}$$

Since the bilinear form $a(\cdot, \cdot)$ is symmetric on $V \times V$ and $\mathcal{A}$ is a unbounded operator we can show that λ_j are real and positive and $\lim_{j\to\infty} \lambda_j = \infty$.

Remark 1.2.1 *An operator $\mathcal{A}$ is positivity preserving if $\mathcal{A}f \geq 0$ when $f \geq 0$. We note that by the maximum principle, $(\mu\mathcal{I} + \mathcal{A})^{-1}$ is a positivity preserving operator for $\mu \geq 0$ and the formula (1.5) shows that $\mathcal{A}^{-\alpha}$ is also. In many applications, it is important that the discrete approximations share this property.*

1.2.2 Sub-diffusion-reaction problems

Another possible model of sub-diffusion reaction is give by the operator equation: find $u \in V$ s.t.

$$\mathcal{A}^\alpha u + bu = f \quad \text{where} \quad b = const \geq 0. \tag{1.7}$$

1.2.3 Transient sub-diffusion-reaction problems

For time dependent problems one can consider: find $u(t) \in V$ for $t \in (0, t_{max}]$ such that

$$\frac{\partial u(t)}{\partial t} + \mathcal{A}^\alpha u(t) = f(t) \quad \text{and } u(0) = v, \tag{1.8}$$

with v given initial data, $t_{max} > 0$ is a given number, and the finite dimensional operator (matrix) $\mathbb{A}$ defined by (1.3).

1.3 Semi-discrete approximations of equations with fractional powers of elliptic operators

We study approximations to $u = \mathcal{A}^{-\alpha} f$ defined in terms of finite difference or finite element approximation of the operator T. We shall use the following convention regarding the approximate solutions by the finite element, [19], and finite difference, [52], methods.

The finite element solutions are functions in V_h, an N-dimensional space of continuous piece-wise linear functions over a partition $\mathcal{T}_h$ of the domain. Such functions will be denoted by u_h, v_h, etc. Also we shall denote by $\mathbb{A}$, $\mathbb{I}$, etc operators acting on the elements u_h, θ_h, etc in the finite dimensional space of functions V_h. When a nodal basis of the finite element space is introduced, then the vectors coefficients in this basis are denoted $\widetilde{u}_h$, $\widetilde{v}_h$, etc. Under this convention operator equations in V_h such as $\mathbb{A}u_h = f_h$ will be written as a system of linear algebraic equations $\widetilde{\mathbb{A}}\widetilde{u}_h = \widetilde{f}_h$ in $\mathbb{R}^N$.

In the finite difference case, discrete solutions are vectors in $\mathbb{R}^N$ and are also denoted $\widetilde{u}_h$, $\widetilde{v}_h$, etc. Then the corresponding counterparts of operators action on these vectors are denoted by $\widetilde{\mathbb{A}}, \widetilde{\mathbb{I}}$, etc.

1.3.1 The finite difference approximation

In this case the approximation $\widetilde{u}_h \in \mathbb{R}^N$ of u is given by

$$\widetilde{\mathbb{A}}^\alpha \widetilde{u}_h = \widetilde{\mathcal{I}}_h f := \widetilde{f}_h, \quad \text{or equivalently} \quad \widetilde{u}_h = \widetilde{\mathbb{A}}^{-\alpha} \widetilde{f}_h, \tag{1.9}$$

where $\widetilde{\mathbb{A}}$ is an $N \times N$ symmetric and positive definite matrix coming from a finite difference approximation to the differential operator appearing in (1.1), $\widetilde{u}_h$ is the vector in $\mathbb{R}^N$ of the approximate solution at the interior N grid points, and $\widetilde{\mathcal{I}}_h f := \widetilde{f}_h \in \mathbb{R}^N$ denotes the vector of the values of the data f at the grid points.

Example 1 We first consider the one-dimensional equation (1.1) with variable coefficient, namely, we study the following boundary value problem $-(a(x)u')' = f(x)$, $u(0) = 0$, $u(1) = 0$, for $0 < x < 1$, where $a(x)$ is uniformly positive function on $[0, 1]$. On a uniform mesh $x_i = ih$, $i = 0, \ldots, N+1$, $h = 1/(N+1)$, we consider the three-point approximation of the second derivative

$$(a(x_i)u'(x_i))' \approx \frac{1}{h}\left(a_{i+\frac{1}{2}} \frac{u(x_{i+1}) - u(x_i)}{h} - a_{i-\frac{1}{2}} \frac{u(x_i) - u(x_{i-1})}{h}\right)$$

Here $a_{i-\frac{1}{2}} = a(x_i - h/2)$ or $a_{i-\frac{1}{2}} = \frac{1}{h}\int_{x_{i-1}}^{x_i} a(x)dx$. Note that the former is the standard finite difference approximation obtained from the balanced method (see, e.g. [52, pp. 155–157]).

Then the finite difference approximation of the differential equation $-(a(x)u'(x))' = f(x)$ is given by the matrix equation (1.9) with

$$\widetilde{\mathbb{A}} = \frac{1}{h^2}\begin{bmatrix} a_{\frac{1}{2}} + a_{\frac{3}{2}} & -a_{\frac{3}{2}} & & & \\ -a_{\frac{3}{2}} & a_{\frac{3}{2}} + a_{\frac{5}{2}} & -a_{\frac{5}{2}} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & -a_{i-\frac{1}{2}} & a_{i-\frac{1}{2}} + a_{i+\frac{1}{2}} & a_{i+\frac{1}{2}} & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -a_{N-\frac{1}{2}} & a_{N-\frac{1}{2}} + a_{N+\frac{1}{2}} \end{bmatrix}, \tag{1.10}$$

The eigenvalues λ_i of the matrix $\mathbb{A}$ are all real and positive and satisfy

$$4\pi^2 \min_x a(x) \le \lambda_i \le 4 \max_x a(x)/h^2, \quad i = 1, \dots, N.$$

1.3.2 The finite element approximation

The approximation in the finite element case is defined in terms of a conforming finite dimensional space $V_h \subset V$ of piece-wise linear functions over a quasi-uniform partition $\mathcal{T}_h$ of Ω into triangles or tetrahedrons. Note that the construction (1.5) of negative fractional powers carries over to the finite dimensional case, replacing V and X by V_h with $a(\cdot,\cdot)$ and $(\cdot,\cdot)$ unchanged.

The discrete operator $\mathbb{A}$ is defined to be the inverse of $T_h : V_h \to V_h$ with $T_h g_h := v_h$ where $v_h \in V_h$ is the unique solution to

$$a(v_h, \theta_h) = (g_h, \theta_h), \quad \text{for all } \theta_h \in V_h. \tag{1.11}$$

The finite element approximation $u_h \in V_h$ of u is then given by

$$\mathbb{A}^\alpha u_h = \pi_h f, \quad \text{or equivalently} \quad u_h = \mathbb{A}^{-\alpha}\pi_h f := \mathbb{A}^{-\alpha} f_h, \tag{1.12}$$

where π_h denotes the $L^2(\Omega)$ projection into V_h. In this case, N denotes the dimension of the space V_h and equals the number of (interior) degrees of freedom. The operator $\mathbb{A}$ in the finite element case is a map of V_h into V_h so that $\mathbb{A}v_h := g_h$, where $g_h \in V_h$ is the unique solution to

$$(g_h, \theta_h) = a(v_h, \theta_h), \quad \text{for all } \theta_h \in V_h. \tag{1.13}$$

Let $\{\phi_j\}$ denote the standard "nodal" basis of V_h. In terms of this basis

$$\mathbb{A} \text{ corresponds to the matrix } \widetilde{\mathbb{A}} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{S}}, \quad \text{where } \widetilde{\mathbb{S}}_{i,j} = a(\phi_i, \phi_j), \quad \widetilde{\mathbb{M}}_{i,j} = (\phi_i, \phi_j). \tag{1.14}$$

In the terminology of the finite element method, $\widetilde{\mathbb{M}}$ and $\widetilde{\mathbb{S}}$ are the mass (consistent mass) and stiffness matrices, respectively.

Obviously, if $\theta = \mathbb{A}\eta$ and $\widetilde{\theta}, \widetilde{\eta} \in \mathbb{R}^N$ are the coefficient vectors corresponding to $\theta, \eta \in V_h$, then $\widetilde{\theta} = \widetilde{\mathbb{A}}\widetilde{\eta}$. Now, for the coefficient vector $\widetilde{f}_h$ corresponding to $f_h = \pi_h f$ we have $\widetilde{f}_h = \widetilde{\mathbb{M}}^{-1}\widetilde{F}$, where $\widetilde{F}$ is the vector with entries

$$\widetilde{F}_j = (f, \phi_j), \qquad \text{for } j = 1, 2, \dots, N.$$

Then using vector notation so that $\widetilde{u}_h$ is the coefficient vector representing the solution u_h through the nodal basis, we can write the finite element approximation of (1.1) in the form of an algebraic system

$$\widetilde{\mathbb{A}}\widetilde{u}_h = \widetilde{\mathbb{M}}^{-1}\widetilde{F} \text{ which implies } \widetilde{\mathbb{S}}\widetilde{u}_h = \widetilde{F}. \tag{1.15}$$

Consequently, the finite element approximation of the sub-diffusion problem (1.12) becomes

$$\widetilde{\mathbb{M}}\widetilde{\mathbb{A}}^\alpha \widetilde{u}_h = \widetilde{F} \quad \text{or} \quad \widetilde{u}_h = \widetilde{\mathbb{A}}^{-\alpha}\widetilde{\mathbb{M}}^{-1}\widetilde{F}. \tag{1.16}$$

1.3.3 The lumped mass finite element approximation

We shall also introduce the finite element method with "mass lumping" for two reasons. First, it leads to positivity preserving fully discrete methods. Second, it is well known that on uniform meshes lumped mass schemes for linear elements are equivalent to the simplest finite difference approximations. This will be used to study the convergence of the finite difference method for solving the problem (1.4), an outstanding issue in this area.

We introduce the lumped mass (discrete) inner product $(\cdot,\cdot)_h$ on V_h in the following way (see, e.g. [58, pp. 239–242]) for d-simplexes in $\mathbb{R}^d$:

$$(z,v)_h = \frac{1}{d+1}\sum_{\tau\in\mathcal{T}_h}\sum_{i=1}^{d+1}|\tau|z(P_i)v(P_i) \quad \text{and} \quad \widetilde{\mathbb{M}}_h = \{(\phi_i,\phi_k)_h\}_{i,k}^N. \tag{1.17}$$

Here $P_1,\dots,P_{d+1}$ are the vertexes of the d-simplex τ and $|\tau|$ is its d-dimensional measure. The matrix $\widetilde{\mathbb{M}}_h$ is called lumped mass matrix. Simply, the "lumped mass" inner product is defined by replacing the integrals determining the finite element mass matrix by local quadrature approximation, specifically, the quadrature defined by summing values at the vertices of a triangle weighted by the area of the triangle.

In this case, we define $\mathbb{A}$ by $\mathbb{A}v_h := g_h$ where $g_h \in V_h$ is the unique solution to

$$(g_h,\theta_h)_h = a(v_h,\theta_h), \qquad \text{for all } \theta_h \in V_h \tag{1.18}$$

so that

$$\mathbb{A} \text{ corresponds to the matrix } \widetilde{\mathbb{A}} = \widetilde{\mathbb{M}}_h^{-1}\widetilde{\mathbb{S}}, \quad \widetilde{\mathbb{M}}_h = \{(\phi_i,\phi_k)_h\}_{i,k}^N. \tag{1.19}$$

Here $\widetilde{\mathbb{M}}_h$ is the lumped mass matrix which is diagonal with positive entries. We also replace π_h by $\mathcal{I}_h$ so that the lumped mass semi-discrete approximation is given by

$$u_h = \mathbb{A}^{-\alpha}\mathcal{I}_h f := \mathbb{A}^{-\alpha} f_h \quad \text{or} \quad \widetilde{u}_h = \widetilde{\mathbb{A}}^{-\alpha}\widetilde{F}. \tag{1.20}$$

Here $\widetilde{F}$ is the coefficient vector in the representation of the function $\mathcal{I}_h f$ with respect to the nodal basis in V_h. We shall call $\widetilde{u}_h$ in (1.9) and u_h in (1.12) and (1.20) *semi-discrete approximations* of u.

1.3.4 Discretization of sub-diffusion-reaction problem (1.7)

If we use (1.9), a finite difference approximation $\widetilde{\mathbb{A}}$ of the operator $\mathcal{A}$, then the corresponding discrete problem is: find $\widetilde{u}_h \in \mathbb{R}^N$ such that

$$\left(\widetilde{\mathbb{A}}^{\alpha} + b\widetilde{\mathbb{I}}\right)\widetilde{u}_h = \widetilde{f}_h.$$

In a similar way one can introduce the corresponding finite element discretizations of the problem (1.7):

$$\left(\mathbb{A}^{\alpha} + b\mathbb{I}\right)u_h = f_h, \tag{1.21}$$

where $\mathbb{A}$ is defined by (1.13) and $\mathbb{I}: V_h \to V_h$ is the identity operator in V_h. Using consistent mass matrix evaluation the operator $\mathbb{I}$ has matrix representation $\widetilde{\mathbb{M}}$ defined by (1.14), while using lumped mass evaluation, the operator $\mathbb{I}$ is represented by the lumped mass matrix $\widetilde{\mathbb{M}}_h$ defined by (1.17).

By using these matrix representations this equation can be written as a system of linear algebraic equations in $\mathbb{R}^N$. Taking into account the matrix representation of the operator $\mathbb{A}$ we get the following systems, corresponding to the consistent mass and lumped mass finite elements approximations of the L^2-inner product in V_h:

$$\widetilde{\mathbb{M}}\left(\widetilde{\mathbb{A}}^{\alpha} + b\widetilde{\mathbb{I}}\right)\widetilde{u}_h = \widetilde{F} \quad \text{or} \quad \widetilde{\mathbb{M}}_h\left(\widetilde{\mathbb{A}}_h^{\alpha} + b\widetilde{\mathbb{I}}\right)\widetilde{u}_h = \widetilde{f}_h.$$

1.3.5 Time-dependent problems

Similarly, the discretization of the time-dependent problem (1.8) with implicit Euler approximation in time, for $t_n = n\tau$, $n = 1, 2, \dots, M$, $\tau = t_{max}/M$ and $u_h^n \in V_h$ an approximation of $u(t_n)$, will lead to the operator equation

$$\left(\frac{1}{\tau}\mathbb{I} + \mathbb{A}^\alpha\right) u_h^n = \frac{1}{\tau} u_h^{n-1} + f_h^n, \quad n = 1, \dots, M, \tag{1.22}$$

where f_h^n is the L^2-projection of $f(t_n)$ on V_h. Denoting by $v_h^n = \frac{1}{\tau} u_h^{n-1} + f_h^n$, we have the following representation of the solution u_h^n:

$$u_h^n = \left(\frac{1}{\tau}\mathbb{I} + \mathbb{A}^\alpha\right)^{-1} v_h^n. \tag{1.23}$$

Having in mind the matrix representations (1.14) (for the consistent mass FEM), (1.17) (for the lumped-mass FEM), and (1.19) of the operator $\mathbb{A} : V_h \to V_h$ we get the following systems of algebraic equations:

$$\left(\frac{1}{\tau}\widetilde{\mathbb{M}} + \widetilde{\mathbb{S}}^\alpha\right) \widetilde{u}_h^n = \frac{1}{\tau}\widetilde{\mathbb{M}}\widetilde{u}_h^{n-1} + \widetilde{F}^n \quad \text{and} \quad \left(\frac{1}{\tau}\widetilde{\mathbb{M}}_h + \widetilde{\mathbb{S}}^\alpha\right) \widetilde{u}_h^n = \frac{1}{\tau}\widetilde{\mathbb{M}}_h\widetilde{u}_h^{n-1} + \widetilde{F}^n,$$

for the standard and lumped mass FEM, correspondingly.

1.4 Brief review of numerical methods for equations with fractional elliptic operators

We note that computing the solutions of (1.12), (1.16), (1.23) involves inverting fractional powers of elliptic operators or their shifts. This is a computationally intensive task and the aim of this handbook is to provide a methodology that results in fast and efficient methods. Further in the handbook these are called fully discrete schemes.

Due to the serious interest of the computational mathematics and physics communities in modeling and simulations involving fractional powers of elliptic operators, a number of approaches and algorithms has been developed, studied and tested on various problems, [1, 46, 47, 31]. We survey some of these approaches by splitting them into four basic categories. These are methods based on:

1. An extension of the problem from $\Omega \subset \mathbb{R}^d$ to a problem in $\Omega \times (0, \infty) \subset \mathbb{R}^{d+1}$, see, e.g. [11]. Nochetto and co-authors in [46, 47] developed efficient computational method based on finite element discretization of the extended problem and subsequent use of multi-grid technique. The main deficiency of the method is that instead of problem in $\mathbb{R}^d$ one needs to work in a domain in one dimension higher which adds to the complexity of the developed algorithms.
2. Reformulation of the problem as a pseudo-parabolic on the cylinder $(0, 1) \times \Omega$ by adding a time variable $t \in (0, 1)$. Such methods were proposed, developed, and tested by Vabishchevich in [60, 61]. As shown in the numerical experiments in [31], while using uniform time stepping, this method is very slow. However, the improvement made in [13, 18] makes the method quite competitive.
3. Approximation of the Dunford-Taylor integral representation of the solution of equations involving fractional powers of elliptic operators, proposed in the pioneering paper of Bonito and Pasciak [9]. Further the idea was extended and augmented in various directions in [7, 8, 10]. These methods use exponentially convergent sinc quadratures and are the most reliable and accurate in the existing literature. Similar flavor has the previous work (based on approach from the numerical linear algebra) on restarted Arnoldi methods for more general matrix functions, see e.g., [20].
4. Best uniform rational approximation of the function t^α on $[0, 1]$, proposed in [25, 29], and further developed in [26, 27, 28] and called BURA methods. We note that t^α, $0 < \alpha < 1$ has derivatives that are singular at the point $t = 0$. Therefore the best uniform rational approximation will have poles that are real, negative, and concentrated at $t = 0$, see [54]. This also means that any "good" approximation of the Balakrishnan

formula (1.5) should have quadrature nodes that are concentrated at $t = 0$. This has been achieved by sinc-quadratures in [7, 8, 10]. Another way to achieve "good" approximation has been obtained recently by Vabishchevich in [62] by a proper change of the variable in the integral (1.5) to the interval $[0, 1]$ and application of Simpson quadrature.

As shown recently in [31], these methods, though entirely different, are interrelated and all seem to involve certain rational approximation of the fractional powers of the underlying elliptic operator. As such, from mathematical point of view, those based on the best uniform rational approximation should be the best. However, one should realize that BURA methods involve application of the Remez method of finding the best uniform rational approximation, [41, 16], a numerical algorithm for solving certain min-max problem that is highly nonlinear and sensitive to the precision of the computer arithmetic. For example, in [63] the errors of the best uniform rational approximation of t^α for six values of $\alpha \in [0, 1]$ are reported for degree $k \le 30$ by using computer arithmetic with 200 significant digits.

1.5 Fully discrete schemes for equations with fractional elliptic operators

1.5.1 Explicit representation of the solution of $u_h = \mathbb{A}^{-\alpha} f_h$

Now consider the spectral properties of the operator $\mathbb{A}$:

$$\mathbb{A}\psi_j = \lambda_j \psi_j \quad \text{or in matrix form} \quad \widetilde{\mathbb{A}}\psi_j = \lambda_j \psi_j, \quad j = 1, \dots, N.$$

Note that if $\widetilde{\mathbb{A}}$ is defined from the finite difference approximation, then this results in a standard matrix eigenvalue problem. In case of finite element approximation (1.14) or (1.19) this results in corresponding generalized eigenvalue problems

$$\widetilde{\mathbb{S}}\psi_j = \lambda_j \widetilde{\mathbb{M}}\psi_j \quad \text{or} \quad \widetilde{\mathbb{S}}\psi_j = \lambda_j \widetilde{\mathbb{M}}_h \psi_j, \quad j = 1, \dots, N. \tag{1.24}$$

Using the eigenvalues and the eigenfunctions we have explicit representation of the solution of the operator equation:

$$u_h = \sum_{j=1}^{N} \lambda_j^{-\alpha} (f_h, \psi_j)\psi_j. \tag{1.25}$$

This representation can be used as a direct method for solving the equation $u_h = \mathbb{A}^{-\alpha} f_h$. Moreover, using FFT-like technique, in the cases when possible (rectangular domain and constant coefficients), this could be an efficient numerical method. However, this will limit substantially the applicability of such approach.

Similarly, the solution of the problem resulting in time-stepping method (1.23) can be expressed through the eigenvalues and the eigenfunctions we have

$$u_h^n = \sum_{j=1}^{N} \left(\lambda_j^\alpha + \frac{1}{\tau}\right)^{-1} (v_h^n, \psi_j)\psi_j = \sum_{j=1}^{N} \lambda_j^{-\alpha} \left(1 + \frac{1}{\tau}\lambda_j^{-\alpha}\right)^{-1} (v_h^n, \psi_j)\psi_j. \tag{1.26}$$

1.5.2 The idea of the fully discrete schemes

To explain our approach we consider $P_k(t)$, the polynomial of degree k that approximates t^α on the interval $[1/\lambda_N, 1/\lambda_1]$ in the maximum norm. Then the vector w_h defined by

$$w_h = \sum_{j=1}^{N} P_k(\lambda_j^{-1})(f_h, \psi_j)\psi_j \quad \text{is an approximation of} \quad u_h = \sum_{j=1}^{N} \lambda_j^{-\alpha} (f_h, \psi_j)\psi_j.$$

Moreover, we can express the error of this approximation through the approximation properties of $P_k(t)$ [21]:

$$\|u_h - w_h\| \le \max_{t=\lambda_1,\dots,\lambda_N} |t^{-\alpha} - P_k(t^{-1})| \|f_h\| = \max_{t=1/\lambda_N,\dots,1/\lambda_1} |t^\alpha - P_k(t)| \|f_h\|.$$

Once we have the polynomial $P_k(t)$ then we can find its roots ξ_i, $i = 1, \ldots, k$, so that we have

$$P_k(t^{-1}) = c_0 \prod_{i=1}^{k} (t^{-1} - \xi_i^{-1})$$

and consequently

$$w_h = c_0 \prod_{i=1}^{k} (\mathbb{A}^{-1} - \xi_i^{-1}\mathbb{I}) f_h := c_0 \prod_{i=1}^{k} w_h^{(i)}.$$

Thus, to find w_h we need to find $w_h^{(i)} = (\mathbb{A}^{-1} - \xi_i^{-1}\mathbb{I}) f_h$, $i = 1, \ldots, k$ which results in solving k systems $\mathbb{A} w_h^{(i)} = (\mathbb{I} - \xi_i^{-1}\mathbb{A}) f_h$.

This idea will produce a computable approximate solution, but will not lead to an efficient method since the required polynomial degree k will depend on the spectral condition number $\kappa(\mathbb{A}) = \lambda_N/\lambda_1$. Namely, the best uniform polynomial approximation of t^α, $t \in [1/\lambda_N, 1/\lambda_1]$, is given by the scaled and shifted Chebyshev polynomial $\widetilde{P}_k(t)$. Then, the error estimate

$$\max_{t=\lambda_1,\ldots,\lambda_N} |t^{-\alpha} - \widetilde{P}_k(t)| < 2 \left(\frac{\sqrt{\kappa^\alpha} + 1}{\sqrt{\kappa^\alpha} - 1} \right)^k$$

holds true, and therefore a polynomial of degree $k \approx 2\kappa^{\frac{\alpha}{2}} \log \frac{2}{\epsilon}$ will be needed to guarantee the the relative error less than ϵ of $\|u_h - w_h\|$. Note, that the matrix $\mathbb{A}$ defined by (1.10) has a condition number $\kappa(\mathbb{A}) = O\left(\max a(x)/\min a(x) h^{-2}\right)$ and the degree of the polynomial of best uniform approximation will grow as $1/h^{\alpha/2}$ as $h \to 0$.

Our aim is to produce a method that involves much smaller k, so we need to solve fewer systems of the type $(\mathbb{A} - \xi_i \mathbb{I})v = f$ with $\xi_i \le 0$.

1.5.3 Fundamental properties of BURA of t^α on $(0, 1]$

Instead of polynomial approximation, we shall seek a rational approximation of the function $t^{-\alpha}$. In order to make the computations uniform and to use the known results for the approximation theory, we first rewrite the solution of the (1.9) in the form

$$u_h = \lambda_1^{-\alpha} (\lambda_1 \mathbb{A}^{-1})^\alpha f_h. \tag{1.27}$$

The scaling by λ_1 maps the eigenvalues of $\lambda_1 \mathbb{A}^{-1}$ to the interval $(\lambda_1/\lambda_N, 1] := (\delta, 1] \subset (0, 1]$. Here $\delta = \lambda_1/\lambda_N$ is a small parameter. Often below we shall take even $\delta = 0$.

Similarly, the solution (1.26) of the time dependent problem after the scaling by λ_1 maps the eigenvalues of $\lambda_1 \mathbb{A}^{-1}$ to the interval $(\lambda_1/\lambda_N, 1] := [\delta, 1] \subset (0, 1]$. Then

$$u_h^n = \sum_{j=1}^{N} \lambda_j^{-\alpha} (1 + b\lambda_j^{-\alpha})^{-1} (v_h^n, \psi_j)\psi_j = \left(b\lambda_1^{-\alpha} (\lambda_1^{-1}\mathbb{A})^{-\alpha} + \mathbb{I} \right)^{-1} (\lambda_1^{-1}\mathbb{A})^{-\alpha} \lambda_1^{-\alpha} v_h^n.$$

Introducing the parameters $q = b\lambda_1^{-1}$ and defining the function $g(t) := g(q, \delta, \alpha; t) = \frac{t^\alpha}{1+qt^\alpha}$ for $t \in (\lambda_1/\lambda_N, 1]$ we get

$$u_h^n = g(\lambda_1 \mathbb{A}^{-1}) \lambda_1^{-\alpha} v_h^n = \sum_{j=1}^{N} g(\lambda_j) \lambda_1^{-\alpha} (v_h^n, \psi_j)\psi_j. \tag{1.28}$$

Now we consider BURA along the diagonal of the Walsh table and take $\mathcal{R}_k$ to be the set of rational functions

$$\mathcal{R}_k = \left\{ r(t) : r(t) = P_k(t)/Q_k(t),\ P_k \in \mathcal{P}_k,\ \text{and}\ \ Q_k \in \mathcal{P}_k,\ \text{monic} \right\}$$

with $\mathcal{P}_k$ set of algebraic polynomials of degree k.

The best *discrete* uniform rational approximation (discrete BURA) of t^α is the rational function $R_{\alpha,k} \in \mathcal{R}_k$ satisfying

$$R_{\alpha,k}(t) := \operatorname*{argmin}_{s(t)\in\mathcal{R}_k} \max_{t\in\{\frac{\lambda_1}{\lambda_N},\frac{\lambda_2}{\lambda_N},\dots,1\}} |s(t)-t^\alpha|. \tag{1.29}$$

Unfortunately, such approximation depends of the knowledge of the eigenvalues, something we would like to avoid. Now we shall show how to avoid in our computation such dependence for both solutions defined by (1.27) and (1.28).

To find a computable approximation to (1.27) we introduce the following best uniform rational approximation (BURA) $r_{\delta,\alpha,k}(t)$ of t^α on $[\delta,1]$

$$r_{\delta,\alpha,k}(t) := \operatorname*{argmin}_{s(t)\in\mathcal{R}_k} \sup_{t\in[\delta,1]} |s(t)-t^\alpha|.$$

Obviously we have

$$\max_{t\in\{\frac{\lambda_1}{\lambda_N},\frac{\lambda_2}{\lambda_N},\dots,1\}} |s(t)-t^\alpha| \le \sup_{t\in[\delta,1]} |s(t)-t^\alpha|.$$

Often, for practical considerations, we would like to get rid of the parameter $\delta = \lambda_1/\lambda_N$ by using the best uniform rational approximation $r_{\alpha,k}(t)$ of t^α on the whole interval $[0,1]$, namely

$$r_{\alpha,k}(t) := \operatorname*{argmin}_{s(t)\in\mathcal{R}_k} \max_{t\in[0,1]} |s(t)-t^\alpha| = \operatorname*{argmin}_{s(t)\in\mathcal{R}_k} \|s(t)-t^\alpha\|_{L^\infty(0,1)}. \tag{1.30}$$

The problem (1.30) has been studied extensively in the past, e.g. [50, 54, 63]. Denoting the error by

$$E_{\alpha,k} := \|r_{\alpha,k}(t)-t^\alpha\|_{L^\infty[0,1]}, \tag{1.31}$$

and applying Theorem 1 of [54] we conclude that there is a constant $C_\alpha > 0$, independent of k, such that

$$E_{\alpha,k} \le C_\alpha e^{-2\pi\sqrt{k\alpha}}. \tag{1.32}$$

Thus, the BURA error converges exponentially to zero as k becomes large.

It is known that the best rational approximation $r_{\alpha,k}(t) = P_k(t)/Q_k(t)$ of t^α for $\alpha \in (0,1)$ is non-degenerate, i.e., the polynomials $P_k(t)$ and $Q_k(t)$ are of full degree k. Denote the roots of $P_k(t)$ and $Q_k(t)$ by $\zeta_1,\dots,\zeta_k$ and $d_1,\dots,d_k$, respectively. It is shown in [50, 55] that the roots are real, interlace and satisfy

$$0 > \zeta_1 > d_1 > \zeta_2 > d_2 > \dots > \zeta_k > d_k. \tag{1.33}$$

We then have

$$r_{\alpha,k}(t) = b\prod_{i=1}^{k} \frac{t-\zeta_i}{t-d_i} \tag{1.34}$$

where, by (1.33) and the fact that $r_{\alpha,k}(t)$ is a best approximation to a non-negative function, $b > 0$ and $P_k(t) > 0$ and $Q_k(t) > 0$ for $t \ge 0$.

Knowing the poles d_i, $i = 1,\dots,k$ we can give an equivalent representation of (1.34) as a sum of partial fractions, namely

$$r_{\alpha,k}(t) = c_0 + \sum_{i=1}^{k} \frac{c_i}{t-d_i} \tag{1.35}$$

where $c_0 > 0$ and $c_i < 0$ for $i = 1,\dots,k$.

Now changing the variable $\xi = 1/t$ in $r_{\alpha,k}(t)$ we get a rational function $\widetilde{r}_{\alpha,k}(\xi)$ defined by

$$\widetilde{r}_{\alpha,k}(\xi) := r_{\alpha,k}(1/t) = \frac{\widetilde{P}(\xi)}{\widetilde{Q}(\xi)}. \tag{1.36}$$

Here $\widetilde{P}_k(\xi) = t^k P_k(t^{-1})$ and $\widetilde{Q}_k(\xi) = t^k Q_k(t^{-1})$ and hence their coefficients are defined by reversing the order of the coefficients in P_k and Q_k appearing in $r_{\alpha,k}(t)$. In addition, (1.33) implies that we have the following properties for the roots of $\widetilde{P}_k$ and $\widetilde{Q}_k$, $\widetilde{d}_i = 1/d_i$ and $\widetilde{\zeta}_i = 1/\zeta_i$, respectively.

$$0 > \widetilde{d}_k > \widetilde{\zeta}_k > \widetilde{d}_{k-1} > \widetilde{\zeta}_{k-1} \cdots > \widetilde{d}_1 > \widetilde{\zeta}_1. \tag{1.37}$$

Remark 1.5.1 *For $\alpha \in (0,1)$,*

$$\widetilde{r}_{\alpha,k}(\xi) = \widetilde{c}_0 + \sum_{i=1}^{k} \widetilde{c}_i/(\xi - \widetilde{d}_i) \tag{1.38}$$

where

$$\widetilde{c}_0 = c_0 - \sum_{i=1}^{k} c_i \widetilde{d}_i = r_{\alpha,k}(0) = E_{\alpha,k} > 0 \quad \textit{with } \widetilde{c}_i = -c_i d_i^{-2} = -c_i \widetilde{d}_i^2 > 0, \; i = 1, \dots, k.$$

Indeed,

$$\widetilde{r}_{\alpha,k}(\xi) = r_{\alpha,k}(1/\xi) = c_0 + \sum_{i=1}^{k} \frac{c_i}{1/\xi - d_i} = c_0 - \sum_{i=1}^{k} c_i d_i^{-1} - \sum_{i=1}^{k} \frac{c_i d_i^{-2}}{\xi - d_i^{-1}}$$

and having in mind that $\widetilde{d}_i = 1/d_i$, we get (1.38).

1.5.4 Fully discrete schemes based on BURA

Now we introduce the *fully discrete* approximations: $w_h \in V_h$ of the finite element approximation $u_h \in V_h$, defined by (1.27), and $\widetilde{w}_h \in \mathbb{R}^N$ of the finite difference approximation $\widetilde{u}_h \in \mathbb{R}^N$ by

$$w_h = \lambda_1^{-\alpha} r_{\alpha,k}(\lambda_1 \mathbb{A}^{-1}) f_h \quad \text{and} \quad \widetilde{w}_h = \lambda_1^{-\alpha} r_{\alpha,k}(\lambda_1 \widetilde{\mathbb{A}}^{-1}) \widetilde{f}_h. \tag{1.39}$$

Here $\mathbb{A}$ and f_h are as in (1.12) or (1.20) and $\widetilde{\mathbb{A}}$ and $\widetilde{f}_h$ are as in (1.9).

In the paper [27], we studied the error of these fully discrete solutions. For the finite element case we obtain the error estimate

$$\|u_h - w_h\| \le \lambda_1^{-\alpha} E_{\alpha,k} \|f_h\| \tag{1.40}$$

with $\|\cdot\|$ denoting the norm in $L^2(\Omega)$. In the finite difference case, we have

$$\|\widetilde{u}_h - \widetilde{w}_h\|_{\ell_2} \le \lambda_1^{-\alpha} E_{\alpha,k} \|\widetilde{f}_h\|_{\ell_2}, \tag{1.41}$$

where the norm $\|\cdot\|_{\ell_2}$ denotes the Euclidean norm in $\mathbb{R}^N$.

1.5.5 BURA approximation of $\frac{t^\alpha}{1+q\,t^\alpha}$ on $[\delta, 1]$

Now we consider the solution (1.28) and introduce the function $g(q, \delta, \alpha; t)$, of the variable t on $[\delta, 1]$, $0 \le \delta < 1$ and two parameters $q \in [0, \infty)$ and $0 < \alpha < 1$:

$$g_{q,\delta,\alpha}(t) := g(q, \delta, \alpha; t) = \frac{t^\alpha}{1 + q\,t^\alpha} \quad \text{on} \quad t \in [\delta, 1].$$

Note that for $q = 1/\tau$ we get the corresponding problem from time-discretization of sub-diffusion equation (1.22). The role of this function is clear from the representation of the solution by (1.28). As before, our goal is to approximate this function using the *best uniform rational approximation* (BURA). To find BURA of $g(q, \delta, \alpha; t)$ we employ Remez algorithm, cf. [41].

Definition 1.5.1 *The best uniform rational approximation $r_{q,\delta,\alpha,k}(t) \in \mathcal{R}_k$ of $g(q,\delta,\alpha;t)$ on $[\delta,1]$, called further (q,δ,α,k)-BURA, is the rational function*

$$r_{q,\delta,\alpha,k}(t) := \operatorname*{argmin}_{s \in \mathcal{R}_k} \|g(q,\delta,\alpha;t) - s(t)\|_{L^\infty[\delta,1]}. \tag{1.42}$$

Then the error-function is denoted by

$$\varepsilon(q,\delta,\alpha,k;t) = r_{q,\delta,\alpha,k}(t) - g(q,\delta,\alpha;t), \tag{1.43}$$

and its L^∞-norm is denoted by

$$E_{q,\delta,\alpha,k} = \sup_{t\in[\delta,1]} |g(q,\delta,\alpha;t) - r_{q,\delta,\alpha,k}(t)| = \|\varepsilon(q,\delta,\alpha,k;t)\|_{L^\infty[\delta,1]}. \tag{1.44}$$

We observe that the zeros and poles of $r_{q,\delta,\alpha,k}$ are again real, nonnegative, and interlacing for all considered choices of the four parameters q,δ,α,k. Furthermore, there seem to be $2k+2$ (the theoretically maximal possible number) points where $\varepsilon(q,\delta,\alpha,k;t)$ achieves its extremal value $\pm E_{q,\delta,\alpha,k}$. However, we are not aware of a theoretical proof for any of the above observations in the general setting $q > 0$ and/or $\delta > 0$, which is not covered by Section 1.5.3.

Obviously, for $\xi = 1/t$, $0 < t < 1$,

$$\widetilde{\varepsilon}(q,\delta,\alpha,k;\xi) := \varepsilon(q,\delta,\alpha,k;t), \quad \xi \in [1,1/\delta] \tag{1.45}$$

we have

$$E_{q,\delta,\alpha,k} = \|\varepsilon(q,\delta,\alpha,k;t)\|_{L^\infty(\delta,1)} = \|\widetilde{\varepsilon}(q,\delta,\alpha,k;\xi)\|_{L^\infty(1,1/\delta)}.$$

1.5.6 URA approximation of $g(q,\delta,\alpha;t) = \frac{t^\alpha}{1+q\,t^\alpha}$ on $t \in [\delta,1]$

Now we shall introduce a rational approximations of function $g(t)$ that has simpler appearance and that is based on the BURA of $t^\alpha = g(0,\delta,\alpha;t)$.

Definition 1.5.2 *The function*

$$\bar{r}_{q,\delta,\alpha,k}(t) := \frac{r_{0,\delta,\alpha,k}(t)}{1 + q\, r_{0,\delta,\alpha,k}(t)} \in \mathcal{R}_k \tag{1.46}$$

is an approximation of $g(q,\delta,\alpha;t)$ on $[\delta,1]$. Then the error-function is defined as

$$\bar{\varepsilon}(q,\delta,\alpha,k;t) = g(q,\delta,\alpha;t) - \bar{r}_{q,\delta,\alpha,k}(t)$$

and

$$\bar{E}_{q,\delta,\alpha,k} = \|g(q,\delta,\alpha;t) - \bar{r}_{q,\delta,\alpha,k}(t)\|_{L^\infty[\delta,1)} = \max_{t\in[\delta,1]} |\bar{\varepsilon}(q,\delta,\alpha,k;t)|.$$

The rational function $\bar{r}_{q,\delta,\alpha,k}(t)$ will be called (q,δ,α,k)-0-URA approximation of $g(q,\delta,\alpha;t)$.

Further, we present this rational function as a sum of partial fractions

$$\bar{r}_{q,\delta,\alpha,k}(t) = \bar{c}_0 + \sum_{i=1}^{k} \frac{\bar{c}_i}{t - \bar{d}_i} \tag{1.47}$$

where $\bar{c}_i > 0$ and $\bar{d}_i$ are the poles, $i = 0,1,\ldots,k$.

As shown in [26, Theorem 2.4] the approximation error $E_{q,\delta,\alpha,k}$ and $\bar{E}_{q,\delta,\alpha,k}$ are related by

$$\bar{E}_{q,\delta,\alpha,k}/(1+q)^2 < E_{q,\delta,\alpha,k} < \bar{E}_{q,\delta,\alpha,k}.$$

The importance of this approximation is that by using corresponding BURA of t^α we reduce the number of the parameters involved.

Remark 1.5.2 *Now consider $q > q_0 > 0$ and take $\bar{r}_{q_0,\delta,\alpha,k}(t)$ as $(q_0, \delta, \alpha, k; t)$-0-URA approximation of $g(q_0, \delta, \alpha; t)$. Then*

$$\bar{r}_{q,\delta,\alpha,k}(t) = \frac{\bar{r}_{q_0,\delta,\alpha,k}(t)}{1 + (q - q_0)\bar{r}_{q_0,\delta,\alpha,k}(t)}$$

$$= \frac{r_{0,\delta,\alpha,k}(t)}{1 + q\, r_{0,\delta,\alpha,k}(t)}\,.$$

Definition 1.5.3 *Let $q = q_0 + q_1$, $q_0, q_1 > 0$, and $r_{q_0,\delta,\alpha,k}(t)$ be (q_0, δ, α, k)-BURA of $g(q_0, \delta, \alpha; t)$. A rational function $\bar{\bar{r}}_{q,\delta,\alpha,k}(t) \in \mathcal{R}_k$ is an uniform approximation of $g(q, \delta, \alpha; t)$ on $[\delta, 1]$, called (q, δ, α, k)-1-URA approximation, and its error $\bar{\bar{E}}_{q,\delta,\alpha,k}$, are defined as:*

$$\bar{\bar{r}}_{q,\delta,\alpha,k}(t) := \frac{r_{q_0,\delta,\alpha,k}(t)}{1 + q_1\, r_{q_0,\delta,\alpha,k}(t)} \tag{1.48}$$

and

$$\bar{\bar{E}}_{q,\delta,\alpha,k} = \|g(q, \delta, \alpha; t) - \bar{\bar{r}}_{q,\delta,\alpha,k}(t)\|_{L^\infty[\delta,1)} = \sup_{t \in [\delta,1]} |\bar{\bar{\varepsilon}}(q, \delta, \alpha, k; t)|\,.$$

Finally, we present this rational function as a sum of partial fractions

$$\bar{\bar{r}}_{q,\delta,\alpha,k}(t) = \bar{\bar{c}}_0 + \sum_{i=1}^{k} \frac{\bar{\bar{c}}_i}{t - \bar{\bar{d}}_i} \tag{1.49}$$

We remark that the (q, δ, α, k)-1-URA approximation gives a possibility to use a previously computed (q_0, δ, α, k)-BURA of $g(q_0, \delta, \alpha; t)$ for a fixed q_0 in order to find an acceptable approximation for $q = q_0 + q_1$ with $q_0, q_1 > 0$.

Chapter 2

Tables

2.1 Description of the data provided by our numerical experiments

We provide all data for the uniform rational approximation of the function $g(q, \delta, \alpha; t)$ for various values of the parameters q, δ, α, k. The Tables and the corresponding files are named according the following encoding:

(a) $q \in \{0,\ 1,\ 100,\ 200,\ 400\}$, coded by $qQQQ$, $QQQ \in \{000, 001, 100, 200, 400\}$, total 5 parameters;

(b) $\delta \in \{0.0,\ 10^{-6},\ 10^{-7},\ 10^{-8}\}$, notation dD, $D \in \{0, 6, 7, 8\}$ – 4 parameters;

(c) $\alpha \in \{0.250,\ 0.500,\ 0.750\}$, notation aAA, $AA \in \{25, 50, 75\}$ – 3 parameters;

(d) $k \in \{3,\ 4,\ 5,\ 6,\ 7,\ 8\}$, notation kK, $k \in \{3, \ldots, 8\}$ – 6 parameters;

(e) q_0 and q_1 go by pairs $(q_0, q_1) = (0, 200)$, $(q_0, q_1) = (100, 100)$, $(q_0, q_1) = (0, 400)$, $(q_0, q_1) = (200, 200)$, total of 4 cases, coded as $\{qq02,\ qq11,\ qq04,\ qq22\}$.

The computational data is presented in a number of Tables that contain:

(a) the errors of (q, δ, α, k)-BURA;

(b) the extreme points of $\varepsilon(q, \delta, \alpha, k; t)$ defined by (1.43);

(c) the poles of $r_{q,\delta,\alpha,k}(t)$ defined by (1.42);

(d) the coefficients of decomposition $\{c_j\}_{j=0}^{k}$ of $r_{q,\delta,\alpha,k}(t)$ as a sum of partial fractions.

(e) the poles of $\bar{r}_{q,\delta,\alpha,k}(t)$ defined by (1.46);

(f) the coefficients of the decomposition $\{\bar{c}_j\}_{j=0}^{k}$ of $\bar{r}_{q,\delta,\alpha,k}(t)$ as a sum of partial fractions.

(g) the poles of $\bar{\bar{r}}_{q,\delta,\alpha,k}(t)$ defined by (1.48);

(h) the coefficients of the decomposition $\{\bar{\bar{c}}_j\}_{j=0}^{k}$ of $\bar{\bar{r}}_{q,\delta,\alpha,k}(t)$ as a sum of partial fractions.

Short description of the type-tables:

(a-b) These tables correspond to characterization of the BURA.

(c-d) These tables correspond to decomposition of BURA as a sum of partial fractions.

(e-f) These tables correspond to decomposition of 0-URA as a sum of partial fractions.

(g-h) These tables correspond to decomposition of 1-URA as a sum of partial fractions.

Table 2.1: Table of Tables and Data-files

Table & File qQQQdDaAAkK.txt	Number of Tables	Rows (q,k)	Cols (d, a)	Folder name	Number of Files
0-head-tabl.txt	(1)	(5*6)	(1+12)	BURA-tabl	360
qQQQdDaAAkK.txt	(q,k)	( )	(d, a)		
BqQQQkK.txt	(5*6)	(2*K+2)	(2+12)	BURA-tabl	360
CqQQQkK.txt	(5*6)	(K)	(2+12)	BURA-dcmp	360
DqQQQkK.txt	(5*6)	(K+1)	(2+12)	BURA-dcmp	360
EqQQQkK.txt	(4*6)	(K)	(2+12)	0URA-dcmp	288
FqQQQkK.txt	(4*6)	(K+1)	(2+12)	0URA-dcmp	288
GqqQQkK.txt	(2*6)	(K)	(2+12)	1URA-dcmp	144
HqqQQkK.txt	(2*6)	(K+1)	(2+12)	1URA-dcmp	144

Total number of Tables is $(1 + 90 + 48 + 24) = 163$.

More descriptions for the tables and files:

(a-b) The data for these tables in files of types (a),(b) correspond to characterization of the BURA. Folder with values of BURA and extreme points has 360 files, and it is named `BURA-tabl`. Only for 5 cases the program did not finished with a solution.

(c-d) The files with normalized coefficients A and B, poles d_j (named $U0(j)$) and coefficients c_j (named $E(j)$) are in the folder named `BURA-dcmp`. One sub-folder more was present (`BURA-dcmp/add/`) with more details about poles and coefficients – its Imaginary parts. Number of files is also 360 and have names `qQQQdDaAAkK`.

(e-f) The files have the same names as in the previous item **(c-d)** and the same structure but folder is other. The folder is named `0URA-dcmp/`. One sub-folder more is given (`0URA-dcmp/add/`) with more details about poles and coefficients – its Imaginary parts. Number of files is 288, because the cases $(0, \delta, \lambda)$-URA coincide with $(0, \delta, \lambda)$-BURA and files with $QQQ = 000$ are not present.

(g-h) The folder with files is `1URA-dcmp/` and sub-folder `1URA-dcmp/add/`. The names of files are `qqQQdDaAAkK` and as the cases `qqQQ = qq02, qq04` are not present because coincide with the cases `qQQQdDaAAkK`, `qQQQ = q200, q400` from `0URA-dcmp`, and number of files is 144 – for `qqQQ = qq11, qq22` only.

2.2 Tables type (a-b) for BURA-errors and BURA-extreme points

It is clear from Table 2.2 that for fixed parameters α, q, and k, the error is increasing, when $\delta \to 0$. However, the differences are not that pronounced, so for practical purposes one can use for all δ the approximations for $\delta = 0$. The significance of using $\delta > 0$ is in the performance of Remez algorithm for computing BURA.

One should realize that BURA-based methods involve Remez method of finding the best uniform rational approximation by solving the highly non-linear min-max problem (1.30). It is well known that Remez algorithm is very sensitive to the precision of the computer arithmetic, cf. [41, 63, 16]. Various techniques for stabilization of the method have been used, mostly by using Tchebyshev orthogonal polynomials, cf. [41, 16]. It seems that to achieve high accuracy one needs to use high arithmetic precision. For example, in [63] the errors of best uniform rational approximation of t^α for six values of $\alpha \in [0, 1]$ are reported for degree $k \le 30$ by using computer arithmetic with 200 significant digits. In short, for $\delta > 0$ the Remez algorithm has substantially better stability and is significantly more reliable.

We also note that in Table 2.2 there are 5 sets of parameters (all of them for $\alpha = 0.25$) for which Remez algorithm dies not provide the needed information. In these cases the convergence in the iterative process for finding the extremal points of BURA for these parameters either does not converge or fail to produce equal absolute values at the extremal points with the desired accuracy.

Table 2.2: The error $E_{q,\delta,\alpha,k}$ of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ for $t \in [\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $k = 3, \ldots, 8$, and $q = 0, 1, 100, 200, 400$

d (δ)	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
q,k	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
000, 3	1.235E-2	2.282E-3	4.041E-4	7.364E-3	2.107E-3	3.993E-4	9.217E-3	2.223E-3	4.032E-4	1.048E-2	2.263E-3	4.039E-4
000, 4	5.566E-3	7.366E-4	9.954E-5	2.270E-3	6.066E-4	9.591E-5	3.263E-3	6.880E-4	9.878E-5	4.082E-3	7.202E-4	9.939E-5
000, 5	2.735E-3	2.690E-4	2.868E-5	6.972E-4	1.823E-4	2.607E-5	1.159E-3	2.309E-4	2.806E-5	1.619E-3	2.550E-4	2.855E-5
000, 6	1.431E-3	1.075E-4	9.252E-6	2.134E-4	5.534E-5	7.548E-6	4.104E-4	8.015E-5	8.773E-6	6.424E-4	9.598E-5	9.146E-6
000, 7	7.865E-4	4.604E-5	3.257E-6	6.522E-5	1.683E-5	2.245E-6	1.450E-4	2.811E-5	2.907E-6	2.545E-4	3.722E-5	3.168E-6
000, 8	4.495E-4	2.085E-5	1.229E-6	1.990E-5	5.121E-6	6.751E-7	5.119E-5	9.881E-6	9.930E-7	1.007E-4	1.460E-5	1.159E-6
001, 3	8.689E-3	1.696E-3	3.064E-4	4.495E-3	1.532E-3	3.018E-4	5.930E-3	1.640E-3	3.055E-4	6.991E-3	1.678E-3	3.062E-4
001, 4	4.077E-3	5.669E-4	7.801E-5	1.386E-3	4.482E-4	7.458E-5	2.109E-3	5.209E-4	7.728E-5	2.756E-3	5.511E-4	7.787E-5
001, 5	2.063E-3	2.122E-4	2.301E-5	4.255E-4	1.353E-4	2.058E-5	7.487E-4	1.769E-4	2.242E-5	1.095E-3	1.989E-4	2.288E-5
001, 6	1.104E-3	8.643E-5	7.556E-6	1.302E-4	4.109E-5	5.999E-6	2.651E-4	6.165E-5	7.101E-6	4.346E-4	7.560E-5	7.453E-6
001, 7	6.174E-4	3.759E-5	2.697E-6	3.979E-5	1.250E-5	1.789E-6	9.366E-5	2.164E-5	2.370E-6	1.721E-4	2.944E-5	2.612E-6
001, 8	3.581E-4	1.723E-5	1.029E-6	1.214E-5	3.803E-6	5.385E-7	3.305E-5	7.609E-6	8.125E-7	6.806E-5	1.156E-5	9.628E-7
100, 3	3.512E-4	1.011E-4	2.243E-5	1.944E-5	5.012E-5	1.997E-5	3.953E-5	7.387E-5	2.183E-5	6.991E-5	8.949E-5	2.231E-5
100, 4	2.012E-4	4.290E-5	7.266E-6	5.995E-6	1.510E-5	5.722E-6	1.404E-5	2.557E-5	6.812E-6	2.756E-5	3.425E-5	7.163E-6
100, 5	1.212E-4	1.943E-5	2.583E-6	1.841E-6	4.587E-6	1.693E-6	4.985E-6	8.962E-6	2.259E-6	1.095E-5	1.339E-5	2.498E-6
100, 6	7.556E-5	9.242E-6	9.847E-7	5.634E-7	1.395E-6	5.071E-7	1.765E-6	3.150E-6	7.717E-7	4.346E-6	5.265E-6	9.187E-7
100, 7	4.831E-5	4.570E-6	3.973E-7	1.721E-7	4.244E-7	1.527E-7	6.237E-7	1.108E-6	2.672E-7	1.721E-6	2.074E-6	3.487E-7
100, 8	- - -	2.336E-6	1.680E-7	5.254E-8	1.291E-7	4.614E-8	2.201E-7	3.896E-7	9.305E-8	6.806E-7	8.170E-7	1.346E-7
200, 3	1.788E-4	5.304E-5	1.218E-5	5.299E-6	1.978E-5	1.019E-5	1.121E-5	3.280E-5	1.164E-5	2.092E-5	4.328E-5	1.206E-5
200, 4	1.033E-4	2.314E-5	4.095E-6	1.634E-6	5.911E-6	2.930E-6	3.976E-6	1.122E-5	3.711E-6	8.212E-6	1.644E-5	4.002E-6
200, 5	6.295E-5	1.080E-5	1.506E-6	5.017E-7	1.793E-6	8.680E-7	1.412E-6	3.919E-6	1.246E-6	3.261E-6	6.406E-6	1.432E-6
200, 6	3.979E-5	5.281E-6	5.917E-7	1.536E-7	5.451E-7	2.601E-7	5.000E-7	1.377E-6	4.280E-7	1.294E-6	2.517E-6	5.354E-7
200, 7	- - -	2.679E-6	2.450E-7	4.693E-8	1.658E-7	7.836E-8	1.767E-7	4.840E-7	1.485E-7	5.125E-7	9.910E-7	2.051E-7
200, 8	- - -	1.401E-6	1.060E-7	1.432E-8	5.045E-8	2.367E-8	6.236E-8	1.702E-7	5.176E-8	2.027E-7	3.904E-7	7.957E-8
400, 3	9.025E-5	2.723E-5	6.417E-6	1.387E-6	7.014E-6	4.891E-6	3.005E-6	1.330E-5	5.957E-6	5.811E-6	1.953E-5	6.311E-6
400, 4	5.239E-5	1.210E-5	2.226E-6	4.276E-7	2.076E-6	1.396E-6	1.065E-6	4.468E-6	1.914E-6	2.273E-6	7.271E-6	2.143E-6
400, 5	3.211E-5	5.778E-6	8.448E-7	1.313E-7	6.284E-7	4.125E-7	3.781E-7	1.551E-6	6.456E-7	9.021E-7	2.807E-6	7.811E-7
400, 6	2.046E-5	2.896E-6	3.416E-7	4.020E-8	1.910E-7	1.235E-7	1.339E-7	5.440E-7	2.222E-7	3.580E-7	1.100E-6	2.956E-7
400, 7	- - -	1.505E-6	1.451E-7	1.228E-8	5.811E-8	3.719E-8	4.732E-8	1.912E-7	7.717E-8	1.418E-7	4.327E-7	1.139E-7
400, 8	- - -	8.049E-7	6.421E-8	3.749E-9	1.768E-8	1.124E-8	1.670E-8	6.726E-8	2.691E-8	5.609E-8	1.704E-7	4.432E-8

Table 2.3: The extreme points of $\varepsilon(q, \delta, \alpha, k; t)$ of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 3}$, and $q = 0$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j	$\alpha = 0.25$	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	1.342E-6	1.037E-4	6.496E-4	1.645E-5	1.354E-4	6.702E-4	6.389E-6	1.134E-4	6.535E-4	3.403E-6	1.067E-4	6.503E-4
3	4.731E-5	1.612E-3	6.907E-3	2.440E-4	1.858E-3	7.007E-3	1.306E-4	1.690E-3	6.926E-3	8.650E-5	1.637E-3	6.911E-3
4	1.011E-3	1.263E-2	3.708E-2	2.682E-3	1.374E-2	3.739E-2	1.819E-3	1.299E-2	3.714E-2	1.423E-3	1.274E-2	3.709E-2
5	1.125E-2	6.819E-2	1.423E-1	2.100E-2	7.163E-2	1.429E-1	1.635E-2	6.931E-2	1.424E-1	1.397E-2	6.855E-2	1.423E-1
6	9.395E-2	2.688E-1	4.016E-1	1.311E-1	2.754E-1	4.024E-1	1.145E-1	2.710E-1	4.017E-1	1.053E-1	2.695E-1	4.016E-1
7	4.887E-1	7.013E-1	7.896E-1	5.472E-1	7.061E-1	7.900E-1	5.229E-1	7.029E-1	7.896E-1	5.082E-1	7.018E-1	7.896E-1
8	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.4: The extreme points of $\varepsilon(q, \delta, \alpha, k; t)$ of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 4}$, and $q = 0$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	5.542E-8	1.080E-5	1.004E-4	5.079E-6	2.285E-5	1.129E-4	1.213E-6	1.427E-5	1.029E-4	4.020E-7	1.186E-5	1.009E-4
3	1.954E-6	1.682E-4	1.073E-3	4.485E-5	2.581E-4	1.137E-3	1.621E-5	1.967E-4	1.086E-3	7.517E-6	1.772E-4	1.076E-3
4	4.176E-5	1.328E-3	5.879E-3	3.420E-4	1.756E-3	6.102E-3	1.655E-4	1.469E-3	5.925E-3	9.807E-5	1.373E-3	5.888E-3
5	4.680E-4	7.419E-3	2.406E-2	2.134E-3	8.969E-3	2.466E-2	1.251E-3	7.944E-3	2.418E-2	8.568E-4	7.590E-3	2.408E-2
6	4.099E-3	3.305E-2	8.003E-2	1.167E-2	3.752E-2	8.132E-2	8.011E-3	3.459E-2	8.030E-2	6.165E-3	3.355E-2	8.008E-2
7	2.769E-2	1.216E-1	2.213E-1	5.551E-2	1.316E-1	2.234E-1	4.315E-2	1.251E-1	2.217E-1	3.625E-2	1.228E-1	2.214E-1
8	1.525E-1	3.585E-1	4.932E-1	2.239E-1	3.729E-1	4.954E-1	1.947E-1	3.636E-1	4.937E-1	1.769E-1	3.602E-1	4.933E-1
9	5.753E-1	7.597E-1	8.327E-1	6.515E-1	7.681E-1	8.336E-1	6.231E-1	7.627E-1	8.329E-1	6.040E-1	7.607E-1	8.327E-1
10	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.5: The extreme points of $\varepsilon(q, \delta, \alpha, k; t)$ of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 5}$, and $q = 0$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	3.229E-9	1.440E-6	1.910E-5	2.756E-6	7.075E-6	2.676E-5	4.753E-7	2.875E-6	2.069E-5	1.049E-7	1.857E-6	1.942E-5
3	1.139E-7	2.242E-5	2.044E-4	1.525E-5	5.885E-5	2.437E-4	4.027E-6	3.373E-5	2.129E-4	1.327E-6	2.601E-5	2.061E-4
4	2.434E-6	1.772E-4	1.124E-3	8.372E-5	3.454E-4	1.265E-3	2.984E-5	2.338E-4	1.155E-3	1.302E-5	1.958E-4	1.130E-3
5	2.728E-5	9.942E-4	4.650E-3	4.129E-4	1.614E-3	5.057E-3	1.827E-4	1.213E-3	4.741E-3	9.609E-5	1.068E-3	4.669E-3
6	2.395E-4	4.496E-3	1.597E-2	1.864E-3	6.420E-3	1.697E-2	9.882E-4	5.200E-3	1.620E-2	6.054E-4	4.735E-3	1.602E-2
7	1.640E-3	1.739E-2	4.777E-2	7.730E-3	2.254E-2	4.988E-2	4.749E-3	1.933E-2	4.825E-2	3.275E-3	1.806E-2	4.787E-2
8	9.687E-3	5.904E-2	1.261E-1	2.972E-2	7.073E-2	1.298E-1	2.079E-2	6.352E-2	1.270E-1	1.587E-2	6.059E-2	1.263E-1
9	4.899E-2	1.748E-1	2.903E-1	1.043E-1	1.959E-1	2.954E-1	8.189E-2	1.830E-1	2.915E-1	6.826E-2	1.776E-1	2.905E-1
10	2.089E-1	4.301E-1	5.608E-1	3.172E-1	4.555E-1	5.655E-1	2.776E-1	4.402E-1	5.619E-1	2.510E-1	4.336E-1	5.610E-1
11	6.373E-1	7.990E-1	8.611E-1	7.260E-1	8.118E-1	8.629E-1	6.969E-1	8.042E-1	8.615E-1	6.754E-1	8.009E-1	8.612E-1
12	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.6: The extreme points of $\varepsilon(q, \delta, \alpha, k; t)$ of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 6}$, and $q = 0$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	2.422E-10	2.299E-7	4.228E-6	1.971E-6	3.488E-6	9.101E-6	2.804E-7	9.260E-7	5.245E-6	4.709E-8	4.131E-7	4.435E-6
3	8.541E-9	3.581E-6	4.524E-5	7.510E-6	2.039E-5	6.892E-5	1.577E-6	8.448E-6	5.075E-5	3.966E-7	5.095E-6	4.640E-5
4	1.826E-7	2.830E-5	2.489E-4	3.133E-5	9.958E-5	3.337E-4	8.760E-6	5.191E-5	2.694E-4	2.926E-6	3.617E-5	2.532E-4
5	2.046E-6	1.589E-4	1.032E-3	1.252E-4	4.124E-4	1.281E-3	4.359E-5	2.499E-4	1.094E-3	1.787E-5	1.905E-4	1.046E-3
6	1.797E-5	7.200E-4	3.569E-3	4.734E-4	1.505E-3	4.205E-3	1.984E-4	1.018E-3	3.729E-3	9.646E-5	8.267E-4	3.604E-3
7	1.232E-4	2.806E-3	1.086E-2	1.692E-3	4.975E-3	1.231E-2	8.304E-4	3.666E-3	1.123E-2	4.636E-4	3.120E-3	1.094E-2
8	7.307E-4	9.740E-3	2.987E-2	5.754E-3	1.513E-2	3.283E-2	3.247E-3	1.195E-2	3.063E-2	2.044E-3	1.056E-2	3.003E-2
9	3.775E-3	3.067E-2	7.502E-2	1.861E-2	4.267E-2	8.039E-2	1.189E-2	3.574E-2	7.641E-2	8.300E-3	3.258E-2	7.532E-2
10	1.759E-2	8.786E-2	1.715E-1	5.707E-2	1.110E-1	1.798E-1	4.088E-2	9.790E-2	1.736E-1	3.132E-2	9.169E-2	1.719E-1
11	7.316E-2	2.247E-1	3.495E-1	1.621E-1	2.607E-1	3.598E-1	1.294E-1	2.406E-1	3.522E-1	1.081E-1	2.308E-1	3.501E-1
12	2.609E-1	4.879E-1	6.126E-1	4.033E-1	5.256E-1	6.210E-1	3.569E-1	5.050E-1	6.148E-1	3.234E-1	4.946E-1	6.130E-1
13	6.836E-1	8.273E-1	8.812E-1	7.801E-1	8.442E-1	8.844E-1	7.523E-1	8.351E-1	8.821E-1	7.303E-1	8.303E-1	8.814E-1
14	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.7: The extreme points of $\varepsilon(q, \delta, \alpha, k; t)$ of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{7}$, and $q = 0$

k=7 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	2.209E-11	4.219E-8	1.051E-6	1.618E-6	2.301E-6	4.350E-6	2.055E-7	4.423E-7	1.716E-6	2.914E-8	1.330E-7	1.189E-6
3	7.789E-10	6.570E-7	1.124E-5	4.642E-6	9.653E-6	2.558E-5	8.260E-7	2.977E-6	1.473E-5	1.699E-7	1.342E-6	1.202E-5
4	1.665E-8	5.192E-6	6.187E-5	1.545E-5	3.892E-5	1.116E-4	3.587E-6	1.568E-5	7.489E-5	9.675E-7	8.661E-6	6.485E-5
5	1.866E-7	2.915E-5	2.568E-4	5.160E-5	1.413E-4	4.016E-4	1.480E-5	6.830E-5	2.963E-4	4.905E-6	4.302E-5	2.660E-4
6	1.639E-6	1.322E-4	8.891E-4	1.673E-4	4.669E-4	1.262E-3	5.749E-5	2.590E-4	9.939E-4	2.269E-5	1.793E-4	9.136E-4
7	1.123E-5	5.158E-4	2.717E-3	5.227E-4	1.427E-3	3.586E-3	2.104E-4	8.830E-4	2.966E-3	9.624E-5	6.570E-4	2.776E-3
8	6.667E-5	1.797E-3	7.543E-3	1.578E-3	4.086E-3	9.407E-3	7.319E-4	2.765E-3	8.087E-3	3.811E-4	2.180E-3	7.672E-3
9	3.450E-4	5.722E-3	1.938E-2	4.604E-3	1.105E-2	2.306E-2	2.427E-3	8.064E-3	2.047E-2	1.416E-3	6.670E-3	1.964E-2
10	1.620E-3	1.687E-2	4.643E-2	1.300E-2	2.831E-2	5.311E-2	7.710E-3	2.208E-2	4.843E-2	4.983E-3	1.902E-2	4.691E-2
11	6.937E-3	4.633E-2	1.038E-1	3.543E-2	6.866E-2	1.147E-1	2.344E-2	5.681E-2	1.071E-1	1.663E-2	5.073E-2	1.046E-1
12	2.744E-2	1.177E-1	2.144E-1	9.230E-2	1.559E-1	2.298E-1	6.778E-2	1.362E-1	2.191E-1	5.246E-2	1.256E-1	2.156E-1
13	9.881E-2	2.705E-1	4.004E-1	2.240E-1	3.233E-1	4.176E-1	1.821E-1	2.967E-1	4.057E-1	1.532E-1	2.819E-1	4.017E-1
14	3.079E-1	5.354E-1	6.535E-1	4.792E-1	5.850E-1	6.666E-1	4.292E-1	5.606E-1	6.576E-1	3.911E-1	5.465E-1	6.544E-1
15	7.194E-1	8.485E-1	8.962E-1	8.202E-1	8.692E-1	9.009E-1	7.945E-1	8.592E-1	8.977E-1	7.731E-1	8.533E-1	8.966E-1
16	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.8: The extreme points of $\varepsilon(q, \delta, \alpha, k; t)$ of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{8}$, and $q = 0$

k=8 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	2.357E-12	8.655E-9	2.865E-7	1.429E-6	1.792E-6	2.706E-6	1.694E-7	2.780E-7	7.363E-7	2.163E-8	6.023E-8	3.800E-7
3	8.310E-11	1.348E-7	3.066E-6	3.315E-6	5.682E-6	1.207E-5	5.222E-7	1.378E-6	5.264E-6	9.249E-8	4.698E-7	3.584E-6
4	1.776E-9	1.065E-6	1.687E-5	9.172E-6	1.910E-5	4.625E-5	1.844E-6	6.137E-6	2.492E-5	4.208E-7	2.669E-6	1.886E-5
5	1.991E-8	5.981E-6	7.002E-5	2.630E-5	6.090E-5	1.534E-4	6.458E-6	2.387E-5	9.440E-5	1.799E-6	1.222E-5	7.622E-5
6	1.748E-7	2.712E-5	2.426E-4	7.474E-5	1.820E-4	4.549E-4	2.181E-5	8.316E-5	3.075E-4	7.208E-6	4.808E-5	2.594E-4
7	1.198E-6	1.058E-4	7.419E-4	2.078E-4	5.124E-4	1.238E-3	7.068E-5	2.654E-4	8.986E-4	2.709E-5	1.687E-4	7.830E-4
8	7.113E-6	3.691E-4	2.065E-3	5.642E-4	1.370E-3	3.141E-3	2.205E-4	7.878E-4	2.415E-3	9.647E-5	5.408E-4	2.158E-3
9	3.682E-5	1.178E-3	5.337E-3	1.495E-3	3.499E-3	7.525E-3	6.637E-4	2.201E-3	6.062E-3	3.269E-4	1.611E-3	5.531E-3
10	1.730E-4	3.493E-3	1.295E-2	3.872E-3	8.577E-3	1.714E-2	1.934E-3	5.836E-3	1.437E-2	1.060E-3	4.514E-3	1.334E-2
11	7.429E-4	9.731E-3	2.975E-2	9.796E-3	2.022E-2	3.724E-2	5.460E-3	1.476E-2	3.232E-2	3.304E-3	1.197E-2	3.045E-2
12	2.971E-3	2.561E-2	6.474E-2	2.419E-2	4.585E-2	7.717E-2	1.495E-2	3.564E-2	6.906E-2	9.912E-3	3.018E-2	6.592E-2
13	1.110E-2	6.360E-2	1.329E-1	5.798E-2	9.936E-2	1.516E-1	3.959E-2	8.190E-2	1.395E-1	2.858E-2	7.211E-2	1.347E-1
14	3.883E-2	1.476E-1	2.545E-1	1.333E-1	2.031E-1	2.787E-1	1.003E-1	1.769E-1	2.631E-1	7.861E-2	1.615E-1	2.568E-1
15	1.250E-1	3.123E-1	4.443E-1	2.860E-1	3.820E-1	4.695E-1	2.368E-1	3.501E-1	4.534E-1	2.012E-1	3.305E-1	4.468E-1
16	3.502E-1	5.750E-1	6.865E-1	5.447E-1	6.352E-1	7.048E-1	4.934E-1	6.085E-1	6.932E-1	4.527E-1	5.915E-1	6.884E-1
17	7.480E-1	8.651E-1	9.079E-1	8.507E-1	8.887E-1	9.141E-1	8.271E-1	8.785E-1	9.102E-1	8.069E-1	8.717E-1	9.085E-1
18	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.9: The extreme points of $\varepsilon(q, \delta, \alpha, k; t)$ of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{3}$, and $q = 1$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	3.498E-7	5.726E-5	4.471E-4	9.698E-6	8.172E-5	4.657E-4	3.021E-6	6.467E-5	4.507E-4	1.317E-6	5.957E-5	4.478E-4
3	1.288E-5	8.925E-4	4.723E-3	1.229E-4	1.083E-3	4.813E-3	5.501E-5	9.531E-4	4.740E-3	3.130E-5	9.117E-4	4.726E-3
4	3.074E-4	7.073E-3	2.520E-2	1.280E-3	7.966E-3	2.549E-2	7.497E-4	7.363E-3	2.526E-2	5.237E-4	7.165E-3	2.521E-2
5	3.893E-3	3.986E-2	9.862E-2	1.037E-2	4.293E-2	9.929E-2	7.153E-3	4.087E-2	9.875E-2	5.591E-3	4.018E-2	9.864E-2
6	4.179E-2	1.777E-1	3.040E-1	7.497E-2	1.853E-1	3.051E-1	5.992E-2	1.802E-1	3.042E-1	5.172E-2	1.785E-1	3.040E-1
7	3.259E-1	5.902E-1	7.089E-1	4.194E-1	5.990E-1	7.097E-1	3.815E-1	5.931E-1	7.090E-1	3.580E-1	5.912E-1	7.089E-1
8	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.10: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t)=t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha=0.25,0.5,0.75$, $\mathbf{k}=\mathbf{4}$, and $q=1$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	1.642E-8	6.397E-6	7.245E-5	3.852E-6	1.617E-5	8.378E-5	7.865E-7	9.141E-6	7.473E-5	2.161E-7	7.229E-6	7.288E-5
3	5.905E-7	9.964E-5	7.724E-4	2.867E-5	1.707E-4	8.306E-4	8.925E-6	1.222E-4	7.843E-4	3.538E-6	1.068E-4	7.747E-4
4	1.327E-5	7.878E-4	4.211E-3	1.987E-4	1.126E-3	4.414E-3	8.338E-5	9.004E-4	4.253E-3	4.310E-5	8.243E-4	4.219E-3
5	1.572E-4	4.430E-3	1.715E-2	1.192E-3	5.677E-3	1.770E-2	6.131E-4	4.857E-3	1.726E-2	3.732E-4	4.569E-3	1.717E-2
6	1.518E-3	2.012E-2	5.725E-2	6.554E-3	2.391E-2	5.848E-2	4.006E-3	2.145E-2	5.751E-2	2.794E-3	2.056E-2	5.730E-2
7	1.168E-2	7.818E-2	1.637E-1	3.299E-2	8.770E-2	1.659E-1	2.321E-2	8.156E-2	1.641E-1	1.796E-2	7.930E-2	1.638E-1
8	8.061E-2	2.607E-1	3.992E-1	1.523E-1	2.784E-1	4.021E-1	1.227E-1	2.671E-1	3.998E-1	1.049E-1	2.629E-1	3.993E-1
9	4.322E-1	6.732E-1	7.723E-1	5.553E-1	6.881E-1	7.740E-1	5.114E-1	6.787E-1	7.726E-1	4.808E-1	6.751E-1	7.723E-1
10	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.11: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t)=t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha=0.25,0.5,0.75$, $\mathbf{k}=\mathbf{5}$, and $q=1$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	1.060E-9	8.967E-7	1.424E-5	2.372E-6	5.717E-6	2.124E-5	3.725E-7	2.076E-6	1.569E-5	7.213E-8	1.233E-6	1.452E-5
3	3.775E-8	1.397E-5	1.522E-4	1.135E-5	4.360E-5	1.879E-4	2.686E-6	2.305E-5	1.600E-4	7.789E-7	1.684E-5	1.537E-4
4	8.273E-7	1.104E-4	8.348E-4	5.672E-5	2.450E-4	9.630E-4	1.800E-5	1.557E-4	8.637E-4	6.937E-6	1.253E-4	8.407E-4
5	9.531E-6	6.199E-4	3.443E-3	2.641E-4	1.114E-3	3.812E-3	1.042E-4	7.959E-4	3.528E-3	4.879E-5	6.794E-4	3.461E-3
6	8.766E-5	2.813E-3	1.178E-2	1.156E-3	4.360E-3	1.269E-2	5.481E-4	3.386E-3	1.199E-2	3.018E-4	3.010E-3	1.182E-2
7	6.370E-4	1.099E-2	3.519E-2	4.766E-3	1.524E-2	3.714E-2	2.636E-3	1.261E-2	3.564E-2	1.651E-3	1.155E-2	3.528E-2
8	4.120E-3	3.828E-2	9.395E-2	1.878E-2	4.855E-2	9.759E-2	1.194E-2	4.229E-2	9.480E-2	8.371E-3	3.969E-2	9.413E-2
9	2.367E-2	1.204E-1	2.253E-1	7.023E-2	1.415E-1	2.310E-1	5.072E-2	1.288E-1	2.267E-1	3.931E-2	1.234E-1	2.256E-1
10	1.242E-1	3.335E-1	4.744E-1	2.410E-1	3.654E-1	4.808E-1	1.980E-1	3.465E-1	4.759E-1	1.695E-1	3.382E-1	4.747E-1
11	5.139E-1	7.299E-1	8.140E-1	6.551E-1	7.515E-1	8.172E-1	6.114E-1	7.389E-1	8.148E-1	5.779E-1	7.332E-1	8.141E-1
12	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.12: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t)=t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha=0.25,0.5,0.75$, $\mathbf{k}=\mathbf{6}$, and $q=1$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	8.636E-11	1.487E-7	3.227E-6	1.807E-6	3.078E-6	7.729E-6	2.433E-7	7.452E-7	4.164E-6	3.764E-8	3.006E-7	3.419E-6
3	3.061E-9	2.316E-6	3.452E-5	6.142E-6	1.650E-5	5.602E-5	1.192E-6	6.319E-6	3.957E-5	2.726E-7	3.550E-6	3.559E-5
4	6.631E-8	1.830E-5	1.897E-4	2.360E-5	7.678E-5	2.663E-4	6.037E-6	3.745E-5	2.086E-4	1.823E-6	2.469E-5	1.938E-4
5	7.541E-7	1.028E-4	7.859E-4	8.907E-5	3.078E-4	1.010E-3	2.823E-5	1.762E-4	8.426E-4	1.047E-5	1.285E-4	7.982E-4
6	6.784E-6	4.661E-4	2.710E-3	3.235E-4	1.098E-3	3.283E-3	1.232E-4	7.070E-4	2.857E-3	5.427E-5	5.532E-4	2.742E-3
7	4.795E-5	1.820E-3	8.221E-3	1.128E-3	3.569E-3	9.526E-3	5.033E-4	2.519E-3	8.561E-3	2.557E-4	2.079E-3	8.295E-3
8	2.978E-4	6.352E-3	2.255E-2	3.804E-3	1.076E-2	2.524E-2	1.956E-3	8.178E-3	2.326E-2	1.127E-3	7.041E-3	2.270E-2
9	1.634E-3	2.027E-2	5.679E-2	1.242E-2	3.041E-2	6.180E-2	7.265E-3	2.460E-2	5.812E-2	4.670E-3	2.193E-2	5.708E-2
10	8.305E-3	5.987E-2	1.320E-1	3.936E-2	8.086E-2	1.403E-1	2.599E-2	6.909E-2	1.343E-1	1.849E-2	6.346E-2	1.325E-1
11	3.909E-2	1.632E-1	2.816E-1	1.193E-1	2.003E-1	2.930E-1	8.867E-2	1.799E-1	2.846E-1	6.945E-2	1.698E-1	2.822E-1
12	1.690E-1	3.959E-1	5.345E-1	3.299E-1	4.438E-1	5.459E-1	2.770E-1	4.181E-1	5.376E-1	2.394E-1	4.048E-1	5.352E-1
13	5.771E-1	7.707E-1	8.433E-1	7.277E-1	7.982E-1	8.486E-1	6.869E-1	7.838E-1	8.447E-1	6.536E-1	7.760E-1	8.436E-1
14	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.13: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t)=t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha=0.25,0.5,0.75$, $\mathbf{k}=\mathbf{7}$, and $q=1$

k=7 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	8.427E-12	2.812E-8	8.172E-7	1.533E-6	2.136E-6	3.909E-6	1.884E-7	3.856E-7	1.433E-6	2.539E-8	1.060E-7	9.458E-7
3	2.980E-10	4.380E-7	8.743E-6	4.034E-6	8.309E-6	2.183E-5	6.776E-7	2.401E-6	1.194E-5	1.299E-7	1.006E-6	9.465E-6
4	6.417E-9	3.461E-6	4.809E-5	1.252E-5	3.197E-5	9.298E-5	2.715E-6	1.213E-5	6.000E-5	6.764E-7	6.307E-6	5.086E-5
5	7.252E-8	1.943E-5	1.995E-4	3.969E-5	1.123E-4	3.297E-4	1.056E-5	5.141E-5	2.356E-4	3.222E-6	3.076E-5	2.080E-4
6	6.454E-7	8.812E-5	6.899E-4	1.236E-4	3.620E-4	1.024E-3	3.923E-5	1.910E-4	7.857E-4	1.423E-5	1.265E-4	7.128E-4
7	4.499E-6	3.440E-4	2.104E-3	3.751E-4	1.085E-3	2.882E-3	1.391E-4	6.415E-4	2.332E-3	5.844E-5	4.594E-4	2.159E-3
8	2.738E-5	1.200E-3	5.828E-3	1.110E-3	3.059E-3	7.496E-3	4.738E-4	1.986E-3	6.326E-3	2.270E-4	1.515E-3	5.948E-3
9	1.463E-4	3.833E-3	1.493E-2	3.210E-3	8.183E-3	1.824E-2	1.558E-3	5.753E-3	1.593E-2	8.383E-4	4.620E-3	1.517E-2
10	7.188E-4	1.138E-2	3.575E-2	9.089E-3	2.090E-2	4.184E-2	4.972E-3	1.574E-2	3.762E-2	2.976E-3	1.321E-2	3.621E-2
11	3.267E-3	3.177E-2	8.044E-2	2.520E-2	5.109E-2	9.068E-2	1.544E-2	4.090E-2	8.362E-2	1.019E-2	3.567E-2	8.122E-2
12	1.406E-2	8.355E-2	1.697E-1	6.809E-2	1.191E-1	1.852E-1	4.658E-2	1.009E-1	1.746E-1	3.375E-2	9.106E-2	1.709E-1
13	5.704E-2	2.048E-1	3.320E-1	1.758E-1	2.606E-1	3.515E-1	1.345E-1	2.328E-1	3.382E-1	1.069E-1	2.171E-1	3.335E-1
14	2.126E-1	4.492E-1	5.831E-1	4.122E-1	5.125E-1	6.008E-1	3.536E-1	4.820E-1	5.888E-1	3.094E-1	4.640E-1	5.846E-1
15	6.271E-1	8.012E-1	8.649E-1	7.809E-1	8.335E-1	8.724E-1	7.441E-1	8.184E-1	8.673E-1	7.126E-1	8.091E-1	8.655E-1
16	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.14: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t)=t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha=0.25,0.5,0.75$, $\mathbf{k}=\mathbf{8}$, and $q=1$

k=8 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	9.514E-13	5.913E-9	2.263E-7	1.379E-6	1.711E-6	2.529E-6	1.601E-7	2.551E-7	6.464E-7	1.980E-8	5.166E-8	3.132E-7
3	3.360E-11	9.209E-8	2.421E-6	2.998E-6	5.106E-6	1.072E-5	4.525E-7	1.177E-6	4.437E-6	7.603E-8	3.761E-7	2.901E-6
4	7.213E-10	7.277E-7	1.332E-5	7.825E-6	1.645E-5	4.001E-5	1.491E-6	5.022E-6	2.065E-5	3.198E-7	2.063E-6	1.516E-5
5	8.123E-9	4.086E-6	5.528E-5	2.142E-5	5.082E-5	1.304E-4	4.952E-6	1.897E-5	7.744E-5	1.290E-6	9.234E-6	6.101E-5
6	7.188E-8	1.853E-5	1.914E-4	5.866E-5	1.482E-4	3.819E-4	1.603E-5	6.467E-5	2.503E-4	4.935E-6	3.571E-5	2.069E-4
7	4.976E-7	7.232E-5	5.849E-4	1.584E-4	4.088E-4	1.028E-3	5.025E-5	2.028E-4	7.271E-4	1.790E-5	1.237E-4	6.229E-4
8	2.996E-6	2.523E-4	1.626E-3	4.203E-4	1.075E-3	2.586E-3	1.528E-4	5.935E-4	1.943E-3	6.207E-5	3.927E-4	1.712E-3
9	1.580E-5	8.055E-4	4.193E-3	1.096E-3	2.708E-3	6.145E-3	4.520E-4	1.640E-3	4.851E-3	2.067E-4	1.161E-3	4.373E-3
10	7.613E-5	2.393E-3	1.015E-2	2.814E-3	6.570E-3	1.389E-2	1.305E-3	4.313E-3	1.144E-2	6.651E-4	3.236E-3	1.051E-2
11	3.379E-4	6.695E-3	2.328E-2	7.115E-3	1.540E-2	3.001E-2	3.685E-3	1.086E-2	2.563E-2	2.077E-3	8.572E-3	2.393E-2
12	1.413E-3	1.778E-2	5.072E-2	1.773E-2	3.496E-2	6.208E-2	1.020E-2	2.629E-2	5.475E-2	6.317E-3	2.170E-2	5.185E-2
13	5.596E-3	4.499E-2	1.051E-1	4.342E-2	7.669E-2	1.228E-1	2.769E-2	6.122E-2	1.115E-1	1.875E-2	5.263E-2	1.069E-1
14	2.125E-2	1.083E-1	2.061E-1	1.036E-1	1.614E-1	2.308E-1	7.324E-2	1.364E-1	2.151E-1	5.415E-2	1.218E-1	2.087E-1
15	7.675E-2	2.443E-1	3.770E-1	2.357E-1	3.197E-1	4.059E-1	1.851E-1	2.855E-1	3.877E-1	1.496E-1	2.645E-1	3.801E-1
16	2.542E-1	4.949E-1	6.232E-1	4.854E-1	5.719E-1	6.474E-1	4.243E-1	5.384E-1	6.323E-1	3.760E-1	5.167E-1	6.258E-1
17	6.672E-1	8.249E-1	8.815E-1	8.206E-1	8.605E-1	8.911E-1	7.878E-1	8.456E-1	8.851E-1	7.588E-1	8.354E-1	8.825E-1
18	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.15: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t)=t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha=0.25,0.5,0.75$, $\mathbf{k}=\mathbf{3}$, and $q=100$

k=3/δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	1.136E-12	2.045E-7	1.336E-5	2.013E-6	3.411E-6	1.999E-5	2.314E-7	8.732E-7	1.473E-5	2.793E-8	3.782E-7	1.363E-5
3	4.845E-11	3.280E-6	1.381E-4	9.434E-6	2.082E-5	1.703E-4	1.307E-6	8.162E-6	1.451E-4	1.933E-7	4.776E-6	1.395E-4
4	1.779E-9	2.874E-5	7.446E-4	6.248E-5	1.191E-4	8.627E-4	1.028E-5	5.675E-5	7.710E-4	1.789E-6	3.783E-5	7.499E-4
5	3.721E-8	2.112E-4	3.403E-3	4.963E-4	6.988E-4	3.824E-3	9.605E-5	3.706E-4	3.498E-3	1.909E-5	2.643E-4	3.422E-3
6	1.281E-6	1.721E-3	1.657E-2	5.355E-3	4.942E-3	1.833E-2	1.324E-3	2.818E-3	1.696E-2	3.195E-4	2.094E-3	1.665E-2
7	4.953E-5	2.233E-2	1.197E-1	7.502E-2	5.447E-2	1.296E-1	2.527E-2	3.408E-2	1.220E-1	7.590E-3	2.644E-2	1.201E-1
8	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.16: The extreme points of $\varepsilon(q, \delta, \alpha, k; t)$ of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 4}$, and $q = 100$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	1.093E-13	3.667E-8	3.029E-6	1.629E-6	2.260E-6	7.384E-6	1.811E-7	4.220E-7	3.935E-6	2.080E-8	1.227E-7	3.215E-6
3	4.275E-12	5.752E-7	3.180E-5	5.177E-6	9.531E-6	5.191E-5	6.855E-7	2.811E-6	3.655E-5	9.535E-8	1.226E-6	3.281E-5
4	1.207E-10	4.676E-6	1.700E-4	2.203E-5	4.080E-5	2.397E-4	3.424E-6	1.534E-5	1.872E-4	5.567E-7	8.128E-6	1.737E-4
5	1.858E-9	2.848E-5	6.970E-4	1.071E-4	1.707E-4	9.056E-4	1.932E-5	7.445E-5	7.495E-4	3.579E-6	4.420E-5	7.084E-4
6	2.981E-8	1.559E-4	2.541E-3	5.870E-4	7.374E-4	3.154E-3	1.253E-4	3.548E-4	2.697E-3	2.679E-5	2.262E-4	2.575E-3
7	4.421E-7	8.860E-4	9.434E-3	3.609E-3	3.530E-3	1.138E-2	9.320E-4	1.836E-3	9.933E-3	2.320E-4	1.230E-3	9.543E-3
8	1.096E-5	6.101E-3	4.033E-2	2.655E-2	2.047E-2	4.745E-2	9.013E-3	1.160E-2	4.218E-2	2.826E-3	8.158E-3	4.073E-2
9	3.599E-4	6.466E-2	2.318E-1	2.199E-1	1.620E-1	2.589E-1	1.089E-1	1.070E-1	2.391E-1	4.626E-2	8.150E-2	2.334E-1
10	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.17: The extreme points of $\varepsilon(q, \delta, \alpha, k; t)$ of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 5}$, and $q = 100$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	1.357E-14	7.520E-9	7.680E-7	1.432E-6	1.770E-6	3.790E-6	1.556E-7	2.695E-7	1.368E-6	1.730E-8	5.683E-8	8.935E-7
3	5.084E-13	1.173E-7	8.147E-6	3.505E-6	5.584E-6	2.065E-5	4.454E-7	1.311E-6	1.121E-5	5.898E-8	4.340E-7	8.841E-6
4	1.273E-11	9.346E-7	4.407E-5	1.118E-5	1.922E-5	8.592E-5	1.653E-6	5.873E-6	5.521E-5	2.544E-7	2.468E-6	4.667E-5
5	1.706E-10	5.381E-6	1.791E-4	4.021E-5	6.536E-5	2.983E-4	6.829E-6	2.377E-5	2.122E-4	1.195E-6	1.163E-5	1.869E-4
6	2.066E-9	2.600E-5	6.127E-4	1.560E-4	2.215E-4	9.241E-4	3.053E-5	9.108E-5	7.015E-4	6.058E-6	4.936E-5	6.339E-4
7	2.138E-8	1.164E-4	1.928E-3	6.452E-4	7.715E-4	2.727E-3	1.468E-4	3.478E-4	2.159E-3	3.313E-5	2.026E-4	1.984E-3
8	2.481E-7	5.254E-4	6.013E-3	2.878E-3	2.862E-3	8.151E-3	7.803E-4	1.397E-3	6.639E-3	2.050E-4	8.601E-4	6.164E-3
9	3.066E-6	2.607E-3	2.007E-2	1.389E-2	1.177E-2	2.624E-2	4.613E-3	6.244E-3	2.190E-2	1.442E-3	4.052E-3	2.052E-2
10	6.204E-5	1.576E-2	7.685E-2	7.355E-2	5.606E-2	9.593E-2	3.211E-2	3.323E-2	8.265E-2	1.284E-2	2.304E-2	7.827E-2
11	1.763E-3	1.343E-1	3.513E-1	3.850E-1	3.107E-1	3.995E-1	2.461E-1	2.249E-1	3.667E-1	1.386E-1	1.760E-1	3.551E-1
12	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.18: The extreme points of $\varepsilon(q, \delta, \alpha, k; t)$ of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 6}$, and $q = 100$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	1.984E-15	1.700E-9	2.129E-7	1.316E-6	1.519E-6	2.479E-6	1.407E-7	2.032E-7	6.245E-7	1.530E-8	3.414E-8	2.979E-7
3	7.261E-14	2.649E-8	2.269E-6	2.681E-6	3.825E-6	1.030E-5	3.287E-7	7.514E-7	4.221E-6	4.177E-8	1.970E-7	2.735E-6
4	1.708E-12	2.098E-7	1.238E-5	6.953E-6	1.101E-5	3.775E-5	9.818E-7	2.822E-6	1.938E-5	1.440E-7	9.601E-7	1.415E-5
5	2.129E-11	1.187E-6	5.069E-5	2.025E-5	3.206E-5	1.208E-4	3.259E-6	9.905E-6	7.145E-5	5.409E-7	4.016E-6	5.609E-5
6	2.254E-10	5.499E-6	1.726E-4	6.285E-5	9.274E-5	3.479E-4	1.148E-5	3.290E-5	2.269E-4	2.141E-6	1.513E-5	1.870E-4
7	1.962E-9	2.256E-5	5.219E-4	2.040E-4	2.692E-4	9.355E-4	4.241E-5	1.062E-4	6.539E-4	8.876E-6	5.365E-5	5.573E-4
8	1.690E-8	8.709E-5	1.473E-3	6.910E-4	7.985E-4	2.433E-3	1.651E-4	3.435E-4	1.785E-3	3.911E-5	1.864E-4	1.558E-3
9	1.419E-7	3.343E-4	4.083E-3	2.446E-3	2.470E-3	6.356E-3	6.806E-4	1.149E-3	4.832E-3	1.843E-4	6.608E-4	4.287E-3
10	1.374E-6	1.348E-3	1.166E-2	9.106E-3	8.129E-3	1.729E-2	3.017E-3	4.096E-3	1.354E-2	9.555E-4	2.490E-3	1.217E-2
11	1.499E-5	6.049E-3	3.592E-2	3.560E-2	2.905E-2	5.059E-2	1.444E-2	1.610E-2	4.094E-2	5.487E-3	1.042E-2	3.731E-2
12	2.575E-4	3.236E-2	1.236E-1	1.439E-1	1.139E-1	1.615E-1	7.569E-2	7.206E-2	1.370E-1	3.669E-2	5.075E-2	1.273E-1
13	6.286E-3	2.210E-1	4.584E-1	5.251E-1	4.558E-1	5.243E-1	3.906E-1	3.585E-1	4.834E-1	2.657E-1	2.929E-1	4.656E-1
14	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.19: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=7}$, and $q = 100$

k=7 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	3.253E-16	4.158E-10	6.353E-8	1.243E-6	1.373E-6	1.896E-6	1.312E-7	1.692E-7	3.618E-7	1.403E-8	2.440E-8	1.221E-7
3	1.174E-14	6.476E-9	6.786E-7	2.213E-6	2.908E-6	6.137E-6	2.634E-7	4.981E-7	1.936E-6	3.232E-8	1.082E-7	9.858E-7
4	2.664E-13	5.122E-8	3.719E-6	4.909E-6	7.207E-6	1.963E-5	6.659E-7	1.595E-6	8.059E-6	9.354E-8	4.523E-7	4.877E-6
5	3.191E-12	2.883E-7	1.533E-5	1.221E-5	1.849E-5	5.751E-5	1.872E-6	4.951E-6	2.803E-5	2.963E-7	1.692E-6	1.889E-5
6	3.141E-11	1.316E-6	5.249E-5	3.224E-5	4.746E-5	1.550E-4	5.559E-6	1.471E-5	8.551E-5	9.815E-7	5.787E-6	6.206E-5
7	2.490E-10	5.229E-6	1.583E-4	8.838E-5	1.215E-4	3.913E-4	1.711E-5	4.229E-5	2.369E-4	3.360E-6	1.856E-5	1.817E-4
8	1.846E-9	1.896E-5	4.362E-4	2.497E-4	3.130E-4	9.433E-4	5.450E-5	1.197E-4	6.129E-4	1.193E-5	5.729E-5	4.895E-4
9	1.276E-8	6.533E-5	1.135E-3	7.257E-4	8.199E-4	2.221E-3	1.797E-4	3.403E-4	1.522E-3	4.406E-5	1.745E-4	1.253E-3
10	9.100E-8	2.222E-4	2.880E-3	2.173E-3	2.211E-3	5.233E-3	6.168E-4	9.898E-4	3.734E-3	1.711E-4	5.387E-4	3.143E-3
11	6.687E-7	7.742E-4	7.389E-3	6.715E-3	6.210E-3	1.263E-2	2.213E-3	3.002E-3	9.322E-3	7.030E-4	1.728E-3	7.988E-3
12	5.713E-6	2.872E-3	1.978E-2	2.143E-2	1.836E-2	3.185E-2	8.358E-3	9.670E-3	2.433E-2	3.103E-3	5.909E-3	2.120E-2
13	5.613E-5	1.181E-2	5.679E-2	7.015E-2	5.751E-2	8.502E-2	3.323E-2	3.366E-2	6.775E-2	1.478E-2	2.211E-2	6.027E-2
14	8.290E-4	5.620E-2	1.764E-1	2.274E-1	1.884E-1	2.376E-1	1.372E-1	1.273E-1	2.013E-1	7.700E-2	9.241E-2	1.845E-1
15	1.715E-2	3.112E-1	5.467E-1	6.313E-1	5.748E-1	6.243E-1	5.144E-1	4.812E-1	5.808E-1	3.939E-1	4.099E-1	5.582E-1
16	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.20: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=8}$, and $q = 100$

k=8 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1		0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2		1.087E-10	2.018E-8	1.192E-6	1.282E-6	1.596E-6	1.247E-7	1.496E-7	2.507E-7	1.318E-8	1.949E-8	6.184E-8
3		1.692E-9	2.158E-7	1.921E-6	2.374E-6	4.180E-6	2.231E-7	3.661E-7	1.054E-6	2.660E-8	6.855E-8	4.179E-7
4		1.338E-8	1.185E-6	3.773E-6	5.191E-6	1.169E-5	4.941E-7	1.018E-6	3.912E-6	6.684E-8	2.474E-7	1.930E-6
5		7.516E-8	4.902E-6	8.284E-6	1.196E-5	3.128E-5	1.218E-6	2.830E-6	1.267E-5	1.847E-7	8.323E-7	7.181E-6
6		3.416E-7	1.687E-5	1.932E-5	2.785E-5	7.882E-5	3.170E-6	7.646E-6	3.677E-5	5.334E-7	2.607E-6	2.301E-5
7		1.342E-6	5.114E-5	4.667E-5	6.478E-5	1.881E-4	8.519E-6	2.007E-5	9.803E-5	1.583E-6	7.710E-6	6.613E-5
8		4.747E-6	1.408E-4	1.156E-4	1.507E-4	4.293E-4	2.348E-5	5.167E-5	2.444E-4	4.816E-6	2.186E-5	1.748E-4
9		1.562E-5	3.607E-4	2.922E-4	3.529E-4	9.491E-4	6.627E-5	1.319E-4	5.794E-4	1.503E-5	6.041E-5	4.340E-4
10		4.918E-5	8.792E-4	7.536E-4	8.372E-4	2.062E-3	1.919E-4	3.378E-4	1.330E-3	4.838E-5	1.655E-4	1.032E-3
11		1.522E-4	2.088E-3	1.984E-3	2.029E-3	4.476E-3	5.711E-4	8.801E-4	3.017E-3	1.611E-4	4.574E-4	2.407E-3
12		4.757E-4	4.951E-3	5.339E-3	5.055E-3	9.856E-3	1.753E-3	2.360E-3	6.895E-3	5.587E-4	1.297E-3	5.625E-3
13		1.543E-3	1.201E-2	1.468E-2	1.304E-2	2.233E-2	5.564E-3	6.586E-3	1.619E-2	2.026E-3	3.835E-3	1.348E-2
14		5.340E-3	3.047E-2	4.113E-2	3.493E-2	5.254E-2	1.830E-2	1.933E-2	3.965E-2	7.752E-3	1.202E-2	3.373E-2
15		2.026E-2	8.193E-2	1.159E-1	9.694E-2	1.284E-1	6.198E-2	6.000E-2	1.020E-1	3.127E-2	4.042E-2	8.920E-2
16		8.635E-2	2.314E-1	3.136E-1	2.704E-1	3.169E-1	2.096E-1	1.941E-1	2.704E-1	1.315E-1	1.459E-1	2.459E-1
17		3.955E-1	6.172E-1	7.095E-1	6.656E-1	7.008E-1	6.118E-1	5.823E-1	6.585E-1	5.050E-1	5.130E-1	6.333E-1
18		1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.21: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=3}$, and $q = 200$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	7.673E-14	5.630E-8	5.903E-6	1.922E-6	2.490E-6	1.106E-5	2.115E-7	4.920E-7	6.978E-6	2.376E-8	1.572E-7	6.120E-6
3	3.286E-12	9.051E-7	6.095E-5	8.531E-6	1.212E-5	8.523E-5	1.086E-6	3.697E-6	6.648E-5	1.413E-7	1.720E-6	6.209E-5
4	1.222E-10	7.992E-6	3.299E-4	5.497E-5	6.304E-5	4.196E-4	8.108E-6	2.357E-5	3.509E-4	1.199E-6	1.292E-5	3.342E-4
5	2.594E-9	5.980E-5	1.534E-3	4.338E-4	3.601E-4	1.866E-3	7.449E-5	1.501E-4	1.612E-3	1.236E-5	8.935E-5	1.550E-3
6	9.325E-8	5.073E-4	7.786E-3	4.723E-3	2.603E-3	9.290E-3	1.033E-3	1.166E-3	8.143E-3	2.060E-4	7.277E-4	7.861E-3
7	3.798E-6	7.250E-3	6.395E-2	6.800E-2	3.132E-2	7.437E-2	2.038E-2	1.541E-2	6.646E-2	5.061E-3	1.007E-2	6.448E-2
8	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.22: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=4}$, and $q = 200$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	7.638E-15	1.067E-8	1.408E-6	1.583E-6	1.869E-6	4.909E-6	1.712E-7	2.934E-7	2.123E-6	1.875E-8	6.576E-8	1.557E-6
3	2.997E-13	1.675E-7	1.476E-5	4.829E-6	6.463E-6	2.991E-5	6.006E-7	1.551E-6	1.843E-5	7.581E-8	5.396E-7	1.557E-5
4	8.537E-12	1.367E-6	7.883E-5	2.004E-5	2.489E-5	1.305E-4	2.866E-6	7.575E-6	9.211E-5	4.102E-7	3.277E-6	8.181E-5
5	1.327E-10	8.402E-6	3.247E-4	9.640E-5	9.927E-5	4.808E-4	1.584E-5	3.485E-5	3.659E-4	2.539E-6	1.712E-5	3.340E-4
6	2.176E-9	4.693E-5	1.202E-3	5.279E-4	4.250E-4	1.678E-3	1.022E-4	1.640E-4	1.329E-3	1.870E-5	8.699E-5	1.231E-3
7	3.319E-8	2.769E-4	4.627E-3	3.269E-3	2.081E-3	6.240E-3	7.655E-4	8.666E-4	5.062E-3	1.628E-4	4.846E-4	4.726E-3
8	8.824E-7	2.046E-3	2.121E-2	2.446E-2	1.277E-2	2.779E-2	7.595E-3	5.816E-3	2.301E-2	2.047E-3	3.421E-3	2.163E-2
9	3.160E-5	2.539E-2	1.455E-1	2.088E-1	1.146E-1	1.783E-1	9.618E-2	6.171E-2	1.548E-1	3.562E-2	3.961E-2	1.476E-1
10	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.23: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=5}$, and $q = 200$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	9.914E-16	2.321E-9	3.740E-7	1.406E-6	1.571E-6	2.915E-6	1.499E-7	2.141E-7	8.567E-7	1.614E-8	3.740E-8	4.747E-7
3	3.723E-14	3.622E-8	3.963E-6	3.335E-6	4.217E-6	1.358E-5	4.043E-7	8.434E-7	6.325E-6	4.970E-8	2.300E-7	4.515E-6
4	9.388E-13	2.890E-7	2.140E-5	1.040E-5	1.308E-5	5.246E-5	1.440E-6	3.358E-6	2.988E-5	2.004E-7	1.174E-6	2.346E-5
5	1.267E-11	1.671E-6	8.689E-5	3.698E-5	4.200E-5	1.744E-4	5.824E-6	1.270E-5	1.121E-4	9.060E-7	5.198E-6	9.318E-5
6	1.558E-10	8.159E-6	2.985E-4	1.428E-4	1.388E-4	5.293E-4	2.578E-5	4.705E-5	3.670E-4	4.505E-6	2.140E-5	3.158E-4
7	1.643E-9	3.726E-5	9.524E-4	5.914E-4	4.836E-4	1.563E-3	1.239E-4	1.788E-4	1.137E-3	2.452E-5	8.751E-5	9.991E-4
8	1.974E-8	1.742E-4	3.058E-3	2.653E-3	1.833E-3	4.780E-3	6.642E-4	7.327E-4	3.584E-3	1.530E-4	3.795E-4	3.192E-3
9	2.554E-7	9.167E-4	1.076E-2	1.293E-2	7.858E-3	1.614E-2	3.991E-3	3.430E-3	1.243E-2	1.100E-3	1.877E-3	1.119E-2
10	5.733E-6	6.121E-3	4.509E-2	6.968E-2	4.009E-2	6.403E-2	2.859E-2	1.980E-2	5.114E-2	1.023E-2	1.165E-2	4.666E-2
11	1.856E-4	6.433E-2	2.511E-1	3.745E-1	2.533E-1	3.151E-1	2.294E-1	1.579E-1	2.729E-1	1.185E-1	1.069E-1	2.569E-1
12	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.24: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=6}$, and $q = 200$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	1.531E-16	5.552E-10	1.079E-7	1.300E-6	1.405E-6	2.101E-6	1.371E-7	1.751E-7	4.489E-7	1.457E-8	2.590E-8	1.767E-7
3	5.613E-15	8.652E-9	1.150E-6	2.585E-6	3.114E-6	7.534E-6	3.057E-7	5.413E-7	2.665E-6	3.666E-8	1.213E-7	1.518E-6
4	1.327E-13	6.857E-8	6.264E-6	6.573E-6	8.149E-6	2.549E-5	8.809E-7	1.813E-6	1.157E-5	1.190E-7	5.269E-7	7.649E-6
5	1.665E-12	3.887E-7	2.561E-5	1.892E-5	2.233E-5	7.742E-5	2.862E-6	5.913E-6	4.117E-5	4.311E-7	2.049E-6	2.984E-5
6	1.783E-11	1.809E-6	8.712E-5	5.838E-5	6.244E-5	2.157E-4	9.969E-6	1.877E-5	1.278E-4	1.671E-6	7.376E-6	9.847E-5
7	1.574E-10	7.499E-6	2.644E-4	1.892E-4	1.789E-4	5.700E-4	3.667E-5	5.934E-5	3.643E-4	6.863E-6	2.556E-5	2.927E-4
8	1.388E-9	2.951E-5	7.548E-4	6.419E-4	5.331E-4	1.482E-3	1.430E-4	1.919E-4	9.974E-4	3.024E-5	8.864E-5	8.242E-4
9	1.202E-8	1.169E-4	2.142E-3	2.283E-3	1.680E-3	3.941E-3	5.936E-4	6.535E-4	2.751E-3	1.437E-4	3.201E-4	2.317E-3
10	1.226E-7	4.955E-4	6.365E-3	8.560E-3	5.709E-3	1.111E-2	2.664E-3	2.418E-3	8.000E-3	7.587E-4	1.254E-3	6.839E-3
11	1.429E-6	2.398E-3	2.091E-2	3.382E-2	2.138E-2	3.436E-2	1.298E-2	1.006E-2	2.566E-2	4.476E-3	5.587E-3	2.230E-2
12	2.828E-5	1.448E-2	7.967E-2	1.388E-1	8.991E-2	1.196E-1	6.985E-2	4.918E-2	9.442E-2	3.125E-2	3.010E-2	8.407E-2
13	8.278E-4	1.258E-1	3.597E-1	5.167E-1	4.039E-1	4.512E-1	3.750E-1	2.873E-1	3.967E-1	2.418E-1	2.105E-1	3.712E-1
14	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.25: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 7}$, and $q = 200$

k=7 /δ j /α	0.00 0.25	0.00 0.50	0.00 0.75	10^{-6} 0.25	10^{-6} 0.50	10^{-6} 0.75	10^{-7} 0.25	10^{-7} 0.50	10^{-7} 0.75	10^{-8} 0.25	10^{-8} 0.50	10^{-8} 0.75
1		0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2		1.429E-10	3.334E-8	1.232E-6	1.302E-6	1.706E-6	1.287E-7	1.531E-7	2.895E-7	1.354E-8	2.029E-8	8.152E-8
3		2.226E-9	3.560E-7	2.153E-6	2.496E-6	4.868E-6	2.492E-7	3.894E-7	1.351E-6	2.922E-8	7.475E-8	5.986E-7
4		1.761E-8	1.950E-6	4.696E-6	5.685E-6	1.439E-5	6.104E-7	1.122E-6	5.268E-6	8.018E-8	2.789E-7	2.851E-6
5		9.917E-8	8.024E-6	1.154E-5	1.374E-5	3.992E-5	1.681E-6	3.236E-6	1.756E-5	2.452E-7	9.666E-7	1.078E-5
6		4.538E-7	2.744E-5	3.029E-5	3.395E-5	1.035E-4	4.931E-6	9.145E-6	5.196E-5	7.952E-7	3.134E-6	3.483E-5
7		1.811E-6	8.271E-5	8.279E-5	8.516E-5	2.543E-4	1.509E-5	2.550E-5	1.410E-4	2.691E-6	9.711E-6	1.008E-4
8		6.633E-6	2.284E-4	2.338E-4	2.180E-4	6.043E-4	4.799E-5	7.124E-5	3.605E-4	9.511E-6	2.946E-5	2.701E-4
9		2.326E-5	5.995E-4	6.808E-4	5.745E-4	1.422E-3	1.586E-4	2.029E-4	8.953E-4	3.521E-5	8.973E-5	6.941E-4
10		8.135E-5	1.551E-3	2.046E-3	1.575E-3	3.396E-3	5.477E-4	5.999E-4	2.227E-3	1.379E-4	2.813E-4	1.769E-3
11		2.957E-4	4.107E-3	6.357E-3	4.536E-3	8.422E-3	1.983E-3	1.874E-3	5.718E-3	5.744E-4	9.314E-4	4.631E-3
12		1.166E-3	1.154E-2	2.044E-2	1.388E-2	2.213E-2	7.587E-3	6.302E-3	1.559E-2	2.588E-3	3.342E-3	1.287E-2
13		5.231E-3	3.552E-2	6.753E-2	4.554E-2	6.258E-2	3.068E-2	2.329E-2	4.625E-2	1.267E-2	1.340E-2	3.912E-2
14		2.846E-2	1.228E-1	2.217E-1	1.591E-1	1.901E-1	1.296E-1	9.607E-2	1.511E-1	6.863E-2	6.209E-2	1.326E-1
15		2.028E-1	4.575E-1	6.251E-1	5.337E-1	5.664E-1	5.017E-1	4.181E-1	5.081E-1	3.716E-1	3.295E-1	4.759E-1
16		1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.26: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 8}$, and $q = 200$

k=8 /δ j /α	0.00 0.25	0.00 0.50	0.00 0.75	10^{-6} 0.25	10^{-6} 0.50	10^{-6} 0.75	10^{-7} 0.25	10^{-7} 0.50	10^{-7} 0.75	10^{-8} 0.25	10^{-8} 0.50	10^{-8} 0.75
1		0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2		3.908E-11	1.091E-8	1.184E-6	1.234E-6	1.488E-6	1.229E-7	1.395E-7	2.160E-7	1.283E-8	1.718E-8	4.602E-8
3		6.087E-10	1.166E-7	1.881E-6	2.115E-6	3.519E-6	2.136E-7	3.036E-7	7.989E-7	2.456E-8	5.160E-8	2.773E-7
4		4.812E-9	6.404E-7	3.641E-6	4.298E-6	9.141E-6	4.604E-7	7.681E-7	2.765E-6	5.891E-8	1.674E-7	1.219E-6
5		2.704E-8	2.647E-6	7.909E-6	9.366E-6	2.319E-5	1.113E-6	1.991E-6	8.547E-6	1.575E-7	5.220E-7	4.399E-6
6		1.230E-7	9.100E-6	1.832E-5	2.096E-5	5.612E-5	2.861E-6	5.112E-6	2.398E-5	4.452E-7	1.546E-6	1.378E-5
7		4.841E-7	2.754E-5	4.409E-5	4.756E-5	1.299E-4	7.634E-6	1.295E-5	6.226E-5	1.304E-6	4.391E-6	3.894E-5
8		1.721E-6	7.576E-5	1.090E-4	1.092E-4	2.900E-4	2.097E-5	3.262E-5	1.522E-4	3.942E-6	1.212E-5	1.017E-4
9		5.715E-6	1.945E-4	2.757E-4	2.552E-4	6.336E-4	5.920E-5	8.254E-5	3.569E-4	1.229E-5	3.307E-5	2.507E-4
10		1.827E-5	4.772E-4	7.124E-4	6.094E-4	1.375E-3	1.719E-4	2.122E-4	8.182E-4	3.968E-5	9.065E-5	5.970E-4
11		5.794E-5	1.150E-3	1.881E-3	1.498E-3	3.017E-3	5.141E-4	5.609E-4	1.873E-3	1.331E-4	2.539E-4	1.407E-3
12		1.878E-4	2.796E-3	5.083E-3	3.809E-3	6.787E-3	1.590E-3	1.541E-3	4.375E-3	4.669E-4	7.389E-4	3.363E-3
13		6.407E-4	7.047E-3	1.405E-2	1.009E-2	1.587E-2	5.094E-3	4.448E-3	1.062E-2	1.720E-3	2.270E-3	8.348E-3
14		2.374E-3	1.884E-2	3.963E-2	2.796E-2	3.896E-2	1.696E-2	1.364E-2	2.726E-2	6.712E-3	7.489E-3	2.195E-2
15		9.900E-3	5.451E-2	1.126E-1	8.102E-2	1.007E-1	5.832E-2	4.489E-2	7.471E-2	2.778E-2	2.701E-2	6.215E-2
16		4.853E-2	1.713E-1	3.079E-1	2.393E-1	2.683E-1	2.012E-1	1.575E-1	2.167E-1	1.209E-1	1.077E-1	1.892E-1
17		2.850E-1	5.398E-1	7.049E-1	6.346E-1	6.567E-1	6.020E-1	5.311E-1	6.004E-1	4.865E-1	4.431E-1	5.651E-1
18		1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.27: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 3}$, and $q = 400$

k=3/δ j /α	0.00 0.25	0.00 0.50	0.00 0.75	10^{-6} 0.25	10^{-6} 0.50	10^{-6} 0.75	10^{-7} 0.25	10^{-7} 0.50	10^{-7} 0.75	10^{-8} 0.25	10^{-8} 0.50	10^{-8} 0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	4.989E-15	1.484E-8	2.509E-6	1.875E-6	1.997E-6	6.499E-6	2.014E-7	3.192E-7	3.334E-6	2.171E-8	7.570E-8	2.679E-6
3	2.141E-13	2.390E-7	2.589E-5	8.079E-6	7.922E-6	4.368E-5	9.784E-7	1.884E-6	3.008E-5	1.174E-7	6.794E-7	2.679E-5
4	8.012E-12	2.119E-6	1.405E-4	5.128E-5	3.763E-5	2.060E-4	7.090E-6	1.085E-5	1.565E-4	9.439E-7	4.713E-6	1.440E-4
5	1.714E-10	1.603E-5	6.613E-4	4.031E-4	2.089E-4	9.101E-4	6.447E-5	6.660E-5	7.228E-4	9.520E-6	3.170E-5	6.746E-4
6	6.305E-9	1.391E-4	3.454E-3	4.410E-3	1.527E-3	4.643E-3	8.962E-4	5.205E-4	3.749E-3	1.579E-4	2.601E-4	3.518E-3
7	2.638E-7	2.105E-3	3.106E-2	6.442E-2	1.942E-2	4.062E-2	1.799E-2	7.252E-3	3.347E-2	3.945E-3	3.791E-3	3.158E-2
8	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.28: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=4}$, and $q = 400$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	5.056E-16	2.919E-9	6.241E-7	1.560E-6	1.630E-6	3.451E-6	1.659E-7	2.231E-7	1.179E-6	1.768E-8	4.000E-8	7.402E-7
3	1.987E-14	4.586E-8	6.535E-6	4.650E-6	4.751E-6	1.773E-5	5.574E-7	9.366E-7	9.280E-6	6.616E-8	2.597E-7	7.162E-6
4	5.685E-13	3.750E-7	3.487E-5	1.903E-5	1.663E-5	7.202E-5	2.591E-6	4.067E-6	4.476E-5	3.415E-7	1.414E-6	3.720E-5
5	8.888E-12	2.320E-6	1.442E-4	9.104E-5	6.357E-5	2.569E-4	1.414E-5	1.767E-5	1.752E-4	2.065E-6	6.990E-6	1.516E-4
6	1.474E-10	1.312E-5	5.404E-4	4.983E-4	2.699E-4	8.930E-4	9.091E-5	8.159E-5	6.388E-4	1.505E-5	3.486E-5	5.640E-4
7	2.282E-9	7.934E-5	2.139E-3	3.097E-3	1.343E-3	3.397E-3	6.841E-4	4.362E-4	2.493E-3	1.312E-4	1.961E-4	2.224E-3
8	6.301E-8	6.139E-4	1.035E-2	2.339E-2	8.575E-3	1.594E-2	6.883E-3	3.049E-3	1.195E-2	1.680E-3	1.441E-3	1.073E-2
9	2.363E-6	8.453E-3	8.187E-2	2.030E-1	8.414E-2	1.168E-1	8.945E-2	3.572E-2	9.232E-2	3.025E-2	1.844E-2	8.443E-2
10	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.29: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=5}$, and $q = 400$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	6.722E-17	6.647E-10	1.729E-7	1.392E-6	1.437E-6	2.331E-6	1.468E-7	1.794E-7	5.583E-7	1.550E-8	2.689E-8	2.521E-7
3	2.528E-15	1.038E-8	1.830E-6	3.245E-6	3.368E-6	9.153E-6	3.827E-7	5.787E-7	3.602E-6	4.486E-8	1.310E-7	2.256E-6
4	6.397E-14	8.288E-8	9.867E-6	1.000E-5	9.533E-6	3.246E-5	1.332E-6	2.046E-6	1.610E-5	1.735E-7	5.912E-7	1.145E-5
5	8.673E-13	4.808E-7	4.006E-5	3.532E-5	2.913E-5	1.027E-4	5.319E-6	7.229E-6	5.852E-5	7.664E-7	2.437E-6	4.491E-5
6	1.075E-11	2.365E-6	1.381E-4	1.361E-4	9.439E-5	3.044E-4	2.342E-5	2.591E-5	1.888E-4	3.766E-6	9.666E-6	1.517E-4
7	1.145E-10	1.096E-5	4.462E-4	5.641E-4	3.292E-4	8.978E-4	1.126E-4	9.780E-5	5.862E-4	2.044E-5	3.910E-5	4.838E-4
8	1.403E-9	5.255E-5	1.469E-3	2.539E-3	1.269E-3	2.799E-3	6.059E-4	4.067E-4	1.887E-3	1.281E-4	1.717E-4	1.582E-3
9	1.860E-8	2.883E-4	5.400E-3	1.245E-2	5.613E-3	9.844E-3	3.673E-3	1.971E-3	6.821E-3	9.323E-4	8.793E-4	5.787E-3
10	4.429E-7	2.070E-3	2.442E-2	6.765E-2	3.016E-2	4.187E-2	2.674E-2	1.211E-2	3.018E-2	8.894E-3	5.833E-3	2.601E-2
11	1.542E-5	2.547E-2	1.627E-1	3.688E-1	2.107E-1	2.395E-1	2.202E-1	1.100E-1	1.903E-1	1.073E-1	6.169E-2	1.706E-1
12	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.30: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=6}$, and $q = 400$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	0.000E0	0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2	1.042E-17	1.670E-10	5.185E-8	1.291E-6	1.322E-6	1.822E-6	1.351E-7	1.556E-7	3.347E-7	1.415E-8	2.078E-8	1.067E-7
3	3.926E-16	2.603E-9	5.520E-7	2.533E-6	2.635E-6	5.621E-6	2.933E-7	4.080E-7	1.707E-6	3.389E-8	7.887E-8	8.348E-7
4	9.332E-15	2.064E-8	3.005E-6	6.373E-6	6.344E-6	1.740E-5	8.279E-7	1.221E-6	6.913E-6	1.060E-7	3.025E-7	4.056E-6
5	1.174E-13	1.172E-7	1.227E-5	1.823E-5	1.651E-5	4.989E-5	2.657E-6	3.707E-6	2.355E-5	3.755E-7	1.087E-6	1.546E-5
6	1.264E-12	5.475E-7	4.173E-5	5.608E-5	4.494E-5	1.341E-4	9.196E-6	1.129E-5	7.113E-5	1.437E-6	3.723E-6	5.033E-5
7	1.125E-11	2.288E-6	1.271E-4	1.816E-4	1.277E-4	3.482E-4	3.376E-5	3.500E-5	2.000E-4	5.873E-6	1.258E-5	1.488E-4
8	1.006E-10	9.142E-6	3.668E-4	6.167E-4	3.824E-4	9.056E-4	1.318E-4	1.132E-4	5.481E-4	2.588E-5	4.341E-5	4.214E-4
9	8.865E-10	3.715E-5	1.064E-3	2.198E-3	1.225E-3	2.449E-3	5.491E-4	3.916E-4	1.537E-3	1.236E-4	1.589E-4	1.207E-3
10	9.310E-9	1.636E-4	3.278E-3	8.277E-3	4.268E-3	7.130E-3	2.481E-3	1.493E-3	4.622E-3	6.596E-4	6.423E-4	3.689E-3
11	1.128E-7	8.405E-4	1.140E-2	3.288E-2	1.657E-2	2.318E-2	1.221E-2	6.506E-3	1.563E-2	3.954E-3	3.011E-3	1.271E-2
12	2.432E-6	5.596E-3	4.761E-2	1.360E-1	7.355E-2	8.710E-2	6.671E-2	3.418E-2	6.255E-2	2.832E-2	1.762E-2	5.235E-2
13	7.922E-5	5.963E-2	2.611E-1	5.122E-1	3.624E-1	3.795E-1	3.662E-1	2.287E-1	3.113E-1	2.280E-1	1.451E-1	2.779E-1
14	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.31: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=7}$, and $q = 400$

k=7 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1		0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2		4.510E-11	1.658E-8	1.226E-6	1.248E-6	1.555E-6	1.274E-7	1.411E-7	2.379E-7	1.326E-8	1.747E-8	5.592E-8
3		7.025E-10	1.770E-7	2.121E-6	2.200E-6	3.916E-6	2.414E-7	3.141E-7	9.565E-7	2.748E-8	5.370E-8	3.640E-7
4		5.558E-9	9.685E-7	4.582E-6	4.655E-6	1.063E-5	5.806E-7	8.170E-7	3.454E-6	7.297E-8	1.779E-7	1.649E-6
5		3.133E-8	3.982E-6	1.120E-5	1.070E-5	2.780E-5	1.580E-6	2.195E-6	1.095E-5	2.185E-7	5.688E-7	6.040E-6
6		1.436E-7	1.360E-5	2.927E-5	2.563E-5	6.910E-5	4.604E-6	5.923E-6	3.128E-5	6.995E-7	1.745E-6	1.909E-5
7		5.757E-7	4.099E-5	7.989E-5	6.334E-5	1.652E-4	1.404E-5	1.608E-5	8.283E-5	2.351E-6	5.220E-6	5.442E-5
8		2.126E-6	1.136E-4	2.256E-4	1.617E-4	3.877E-4	4.463E-5	4.446E-5	2.092E-4	8.292E-6	1.557E-5	1.449E-4
9		7.567E-6	3.008E-4	6.576E-4	4.292E-4	9.135E-4	1.478E-4	1.271E-4	5.196E-4	3.075E-5	4.738E-5	3.736E-4
10		2.711E-5	7.928E-4	1.981E-3	1.193E-3	2.213E-3	5.120E-4	3.815E-4	1.310E-3	1.210E-4	1.506E-4	9.663E-4
11		1.021E-4	2.164E-3	6.171E-3	3.510E-3	5.636E-3	1.863E-3	1.223E-3	3.455E-3	5.084E-4	5.123E-4	2.602E-3
12		4.238E-4	6.356E-3	1.992E-2	1.104E-2	1.538E-2	7.181E-3	4.263E-3	9.810E-3	2.319E-3	1.916E-3	7.543E-3
13		2.049E-3	2.089E-2	6.615E-2	3.754E-2	4.583E-2	2.931E-2	1.656E-2	3.084E-2	1.154E-2	8.146E-3	2.438E-2
14		1.253E-2	7.991E-2	2.186E-1	1.380E-1	1.503E-1	1.255E-1	7.349E-2	1.100E-1	6.397E-2	4.126E-2	9.083E-2
15		1.128E-1	3.610E-1	6.218E-1	4.998E-1	5.073E-1	4.945E-1	3.623E-1	4.320E-1	3.584E-1	2.578E-1	3.887E-1
16		1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

Table 2.32: The extreme points of $\varepsilon(q,\delta,\alpha,k;t)$ of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=8}$, and $q = 400$

k=8 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1		0.000E0	0.000E0	1.000E-6	1.000E-6	1.000E-6	1.000E-7	1.000E-7	1.000E-7	1.000E-8	1.000E-8	1.000E-8
2		1.290E-11	5.594E-9	1.180E-6	1.196E-6	1.398E-6	1.220E-7	1.317E-7	1.894E-7	1.263E-8	1.547E-8	3.516E-8
3		2.009E-10	5.979E-8	1.859E-6	1.920E-6	2.990E-6	2.084E-7	2.575E-7	6.125E-7	2.340E-8	3.996E-8	1.853E-7
4		1.588E-9	3.281E-7	3.570E-6	3.660E-6	7.186E-6	4.421E-7	5.943E-7	1.960E-6	5.451E-8	1.161E-7	7.667E-7
5		8.928E-9	1.355E-6	7.710E-6	7.599E-6	1.722E-5	1.057E-6	1.440E-6	5.749E-6	1.428E-7	3.348E-7	2.660E-6
6		4.064E-8	4.654E-6	1.779E-5	1.647E-5	3.991E-5	2.697E-6	3.525E-6	1.551E-5	3.987E-7	9.367E-7	8.108E-6
7		1.603E-7	1.407E-5	4.274E-5	3.668E-5	8.945E-5	7.170E-6	8.658E-6	3.910E-5	1.159E-6	2.554E-6	2.243E-5
8		5.723E-7	3.869E-5	1.056E-4	8.360E-5	1.956E-4	1.967E-5	2.143E-5	9.363E-5	3.491E-6	6.875E-6	5.769E-5
9		1.916E-6	9.957E-5	2.671E-4	1.954E-4	4.233E-4	5.552E-5	5.396E-5	2.170E-4	1.088E-5	1.854E-5	1.412E-4
10		6.214E-6	2.462E-4	6.909E-4	4.701E-4	9.205E-4	1.614E-4	1.395E-4	4.972E-4	3.519E-5	5.090E-5	3.365E-4
11		2.015E-5	6.026E-4	1.828E-3	1.170E-3	2.044E-3	4.843E-4	3.739E-4	1.151E-3	1.186E-4	1.445E-4	8.021E-4
12		6.743E-5	1.503E-3	4.950E-3	3.027E-3	4.700E-3	1.504E-3	1.050E-3	2.747E-3	4.189E-4	4.305E-4	1.961E-3
13		2.406E-4	3.927E-3	1.372E-2	8.192E-3	1.133E-2	4.845E-3	3.121E-3	6.890E-3	1.557E-3	1.368E-3	5.034E-3
14		9.475E-4	1.104E-2	3.884E-2	2.331E-2	2.892E-2	1.624E-2	9.939E-3	1.847E-2	6.153E-3	4.723E-3	1.387E-2
15		4.303E-3	3.426E-2	1.108E-1	6.988E-2	7.868E-2	5.633E-2	3.435E-2	5.371E-2	2.585E-2	1.811E-2	4.188E-2
16		2.403E-2	1.196E-1	3.049E-1	2.160E-1	2.253E-1	1.965E-1	1.292E-1	1.698E-1	1.149E-1	7.890E-2	1.402E-1
17		1.807E-1	4.518E-1	7.024E-1	6.088E-1	6.108E-1	5.964E-1	4.844E-1	5.378E-1	4.753E-1	3.766E-1	4.900E-1
18		1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0	1.000E0

2.3 Tables type (c) for BURA-poles

Now we provide the data about the poles of the BURA element $r_{q,\delta,\alpha,k}(t)$:

Table 2.33: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, **k = 3**, and $q = 0$

k=3 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.36E-5	-8.81E-4	-5.45E-3	-9.29E-5	-1.04E-3	-5.53E-3	-4.51E-5	-9.32E-4	-5.46E-3	-2.78E-5	-8.97E-4	-5.45E-3
2	-4.78E-3	-4.93E-2	-1.49E-1	-1.01E-2	-5.24E-2	-1.50E-1	-7.49E-3	-5.03E-2	-1.50E-1	-6.21E-3	-4.97E-2	-1.50E-1
3	-4.14E-1	-1.77E0	-6.09E0	-5.48E-1	-1.82E0	-6.11E0	-4.88E-1	-1.79E0	-6.10E0	-4.55E-1	-1.78E0	-6.10E0

Table 2.34: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, **k = 4**, and $q = 0$

k=4 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-5.61E-7	-9.17E-5	-8.37E-4	-1.95E-5	-1.49E-4	-8.91E-4	-6.35E-6	-1.10E-4	-8.48E-4	-2.67E-6	-9.75E-5	-8.39E-4
2	-1.97E-4	-4.98E-3	-2.11E-2	-1.10E-3	-6.15E-3	-2.17E-2	-6.01E-4	-5.38E-3	-2.13E-2	-3.92E-4	-5.11E-3	-2.12E-2
3	-1.41E-2	-1.01E-1	-2.68E-1	-3.27E-2	-1.12E-1	-2.72E-1	-2.41E-2	-1.05E-1	-2.69E-1	-1.95E-2	-1.03E-1	-2.68E-1
4	-6.27E-1	-2.49E0	-8.40E0	-9.10E-1	-2.62E0	-8.47E0	-7.90E-1	-2.53E0	-8.42E0	-7.20E-1	-2.50E0	-8.41E0

Table 2.35: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, **k = 5**, and $q = 0$

k=5 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-3.27E-8	-1.22E-5	-1.59E-4	-7.21E-6	-3.56E-5	-1.92E-4	-1.77E-6	-1.94E-5	-1.66E-4	-5.28E-7	-1.45E-5	-1.60E-4
2	-1.15E-5	-6.62E-4	-3.97E-3	-2.29E-4	-1.12E-3	-4.33E-3	-9.42E-5	-8.22E-4	-4.05E-3	-4.65E-5	-7.15E-4	-3.98E-3
3	-8.15E-4	-1.28E-2	-4.47E-2	-4.60E-3	-1.70E-2	-4.69E-2	-2.68E-3	-1.44E-2	-4.52E-2	-1.77E-3	-1.33E-2	-4.48E-2
4	-2.81E-2	-1.63E-1	-3.97E-1	-7.13E-2	-1.90E-1	-4.08E-1	-5.27E-2	-1.73E-1	-4.00E-1	-4.21E-2	-1.66E-1	-3.98E-1
5	-8.47E-1	-3.21E0	-1.08E1	-1.35E0	-3.52E0	-1.09E1	-1.15E0	-3.33E0	-1.08E1	-1.03E0	-3.25E0	-1.08E1

Table 2.36: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, **k = 6**, and $q = 0$

k=6 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-2.45E-9	-1.95E-6	-3.52E-5	-3.62E-6	-1.28E-5	-5.49E-5	-7.44E-7	-5.05E-6	-3.98E-5	-1.74E-7	-2.90E-6	-3.62E-5
2	-8.61E-7	-1.06E-4	-8.75E-4	-7.42E-5	-2.93E-4	-1.10E-3	-2.41E-5	-1.72E-4	-9.30E-4	-9.22E-6	-1.28E-4	-8.87E-4
3	-6.11E-5	-2.03E-3	-9.65E-3	-1.06E-3	-3.74E-3	-1.10E-2	-4.90E-4	-2.70E-3	-1.00E-2	-2.60E-4	-2.27E-3	-9.73E-3
4	-2.07E-3	-2.40E-2	-7.40E-2	-1.22E-2	-3.47E-2	-8.01E-2	-7.43E-3	-2.84E-2	-7.56E-2	-4.98E-3	-2.56E-2	-7.44E-2
5	-4.58E-2	-2.30E-1	-5.32E-1	-1.26E-1	-2.85E-1	-5.58E-1	-9.40E-2	-2.53E-1	-5.39E-1	-7.46E-2	-2.39E-1	-5.34E-1
6	-1.07E0	-3.95E0	-1.31E1	-1.86E0	-4.52E0	-1.36E1	-1.57E0	-4.20E0	-1.32E1	-1.38E0	-4.04E0	-1.32E1

Table 2.37: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k}=\mathbf{7}$, and $q=0$

k=7 /δ j /α	0.00 0.25	0.00 0.50	0.00 0.75	10^{-6} 0.25	10^{-6} 0.50	10^{-6} 0.75	10^{-7} 0.25	10^{-7} 0.50	10^{-7} 0.75	10^{-8} 0.25	10^{-8} 0.50	10^{-8} 0.75
1	-2.24E-10	-3.58E-7	-8.75E-6	-2.17E-6	-6.05E-6	-2.05E-5	-3.98E-7	-1.84E-6	-1.17E-5	-7.97E-8	-7.90E-7	-9.40E-6
2	-7.85E-8	-1.94E-5	-2.17E-4	-3.21E-5	-1.03E-4	-3.45E-4	-8.69E-6	-4.79E-5	-2.52E-4	-2.70E-6	-2.94E-5	-2.25E-4
3	-5.57E-6	-3.72E-4	-2.39E-3	-3.42E-4	-1.09E-3	-3.18E-3	-1.30E-4	-6.56E-4	-2.61E-3	-5.65E-5	-4.80E-4	-2.44E-3
4	-1.89E-4	-4.34E-3	-1.77E-2	-3.10E-3	-8.76E-3	-2.13E-2	-1.56E-3	-6.26E-3	-1.88E-2	-8.72E-4	-5.11E-3	-1.80E-2
5	-4.08E-3	-3.80E-2	-1.08E-1	-2.51E-2	-5.94E-2	-1.21E-1	-1.58E-2	-4.79E-2	-1.12E-1	-1.08E-2	-4.21E-2	-1.09E-1
6	-6.66E-2	-3.01E-1	-6.71E-1	-1.97E-1	-3.96E-1	-7.24E-1	-1.48E-1	-3.46E-1	-6.87E-1	-1.17E-1	-3.20E-1	-6.75E-1
7	-1.30E00	-4.69E00	-1.55E01	-2.45E00	-5.64E00	-1.64E01	-2.05E00	-5.15E00	-1.58E01	-1.78E00	-4.88E00	-1.56E01

Table 2.38: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k}=\mathbf{8}$, and $q=0$

k=8 /δ j /α	0.00 0.25	0.00 0.50	0.00 0.75	10^{-6} 0.25	10^{-6} 0.50	10^{-6} 0.75	10^{-7} 0.25	10^{-7} 0.50	10^{-7} 0.75	10^{-8} 0.25	10^{-8} 0.50	10^{-8} 0.75
1	-2.39E-11	-7.35E-8	-2.38E-6	-1.45E-6	-3.45E-6	-9.54E-6	-2.47E-7	-8.67E-7	-4.20E-6	-4.45E-8	-2.86E-7	-2.82E-6
2	-8.37E-9	-3.98E-6	-5.93E-5	-1.69E-5	-4.52E-5	-1.33E-4	-3.97E-6	-1.71E-5	-8.07E-5	-1.04E-6	-8.49E-6	-6.47E-5
3	-5.95E-7	-7.62E-5	-6.50E-4	-1.41E-4	-3.97E-4	-1.10E-3	-4.56E-5	-2.00E-4	-7.91E-4	-1.66E-5	-1.24E-4	-6.87E-4
4	-2.01E-5	-8.89E-4	-4.79E-3	-1.04E-3	-2.79E-3	-6.84E-3	-4.41E-4	-1.72E-3	-5.47E-3	-2.08E-4	-1.24E-3	-4.97E-3
5	-4.34E-4	-7.65E-3	-2.79E-2	-6.99E-3	-1.67E-2	-3.55E-2	-3.73E-3	-1.19E-2	-3.05E-2	-2.17E-3	-9.55E-3	-2.86E-2
6	-6.88E-3	-5.44E-2	-1.44E-1	-4.40E-2	-9.13E-2	-1.69E-1	-2.85E-2	-7.28E-2	-1.53E-1	-1.97E-2	-6.28E-2	-1.47E-1
7	-8.99E-2	-3.75E-1	-8.13E-1	-2.83E-1	-5.23E-1	-9.06E-1	-2.13E-1	-4.51E-1	-8.46E-1	-1.69E-1	-4.10E-1	-8.22E-1
8	-1.53E00	-5.43E00	-1.79E01	-3.11E00	-6.88E00	-1.95E01	-2.58E00	-6.18E00	-1.85E01	-2.23E00	-5.78E00	-1.81E01

Table 2.39: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k}=\mathbf{3}$, and $q=1$

k=3 /δ j /α	0.00 0.25	0.00 0.50	0.00 0.75	10^{-6} 0.25	10^{-6} 0.50	10^{-6} 0.75	10^{-7} 0.25	10^{-7} 0.50	10^{-7} 0.75	10^{-8} 0.25	10^{-8} 0.50	10^{-8} 0.75
1	-3.53E-6	-4.86E-4	-3.78E-3	-4.67E-5	-6.08E-4	-3.87E-3	-1.89E-5	-5.25E-4	-3.80E-3	-9.90E-6	-4.98E-4	-3.79E-3
2	-1.43E-3	-2.61E-2	-9.83E-2	-4.41E-3	-2.84E-2	-9.90E-2	-2.88E-3	-2.69E-2	-9.84E-2	-2.17E-3	-2.64E-2	-9.83E-2
3	-1.49E-1	-5.52E-1	-9.17E-1	-2.37E-1	-5.71E-1	-9.20E-1	-1.97E-1	-5.58E-1	-9.18E-1	-1.76E-1	-5.54E-1	-9.17E-1

Table 2.40: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k}=\mathbf{4}$, and $q=1$

k=4 /δ j /α	0.00 0.25	0.00 0.50	0.00 0.75	10^{-6} 0.25	10^{-6} 0.50	10^{-6} 0.75	10^{-7} 0.25	10^{-7} 0.50	10^{-7} 0.75	10^{-8} 0.25	10^{-8} 0.50	10^{-8} 0.75
1	-1.66E-7	-5.43E-5	-6.06E-4	-1.26E-5	-9.98E-5	-6.56E-4	-3.57E-6	-6.86E-5	-6.16E-4	-1.28E-6	-5.88E-5	-6.08E-4
2	-6.21E-5	-2.94E-3	-1.53E-2	-5.87E-4	-3.85E-3	-1.59E-2	-2.82E-4	-3.25E-3	-1.54E-2	-1.63E-4	-3.04E-3	-1.53E-2
3	-5.06E-3	-5.62E-2	-1.75E-1	-1.68E-2	-6.42E-2	-1.78E-1	-1.11E-2	-5.90E-2	-1.76E-1	-8.30E-3	-5.72E-2	-1.75E-1
4	-2.51E-1	-7.73E-1	-1.13E00	-4.39E-1	-8.23E-1	-1.13E00	-3.61E-1	-7.91E-1	-1.13E00	-3.14E-1	-7.79E-1	-1.13E00

Table 2.41: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{5}$, and $q = 1$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.07E-8	-7.62E-6	-1.19E-4	-5.32E-6	-2.67E-5	-1.49E-4	-1.20E-6	-1.34E-5	-1.25E-4	-3.19E-7	-9.42E-6	-1.20E-4
2	-3.88E-6	-4.12E-4	-2.97E-3	-1.44E-4	-7.74E-4	-3.30E-3	-5.31E-5	-5.39E-4	-3.04E-3	-2.33E-5	-4.55E-4	-2.98E-3
3	-2.94E-4	-7.88E-3	-3.32E-2	-2.67E-3	-1.12E-2	-3.52E-2	-1.40E-3	-9.14E-3	-3.37E-2	-8.39E-4	-8.32E-3	-3.33E-2
4	-1.14E-2	-9.28E-2	-2.55E-1	-4.08E-2	-1.12E-1	-2.62E-1	-2.77E-2	-1.00E-1	-2.57E-1	-2.06E-2	-9.55E-2	-2.55E-1
5	-3.65E-1	-9.93E-1	-1.33E00	-6.96E-1	-1.10E00	-1.35E00	-5.67E-1	-1.04E00	-1.33E00	-4.87E-1	-1.01E00	-1.33E00

Table 2.42: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{6}$, and $q = 1$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-8.74E-10	-1.26E-6	-2.69E-5	-2.89E-6	-1.04E-5	-4.47E-5	-5.63E-7	-3.81E-6	-3.11E-5	-1.22E-7	-2.04E-6	-2.78E-5
2	-3.12E-7	-6.83E-5	-6.69E-4	-5.25E-5	-2.20E-4	-8.69E-4	-1.56E-5	-1.22E-4	-7.20E-4	-5.42E-6	-8.68E-5	-6.80E-4
3	-2.29E-5	-1.31E-3	-7.39E-3	-6.84E-4	-2.67E-3	-8.63E-3	-2.89E-4	-1.85E-3	-7.71E-3	-1.40E-4	-1.51E-3	-7.46E-3
4	-8.29E-4	-1.52E-2	-5.59E-2	-7.59E-3	-2.36E-2	-6.13E-2	-4.23E-3	-1.88E-2	-5.73E-2	-2.61E-3	-1.66E-2	-5.62E-2
5	-2.04E-2	-1.34E-1	-3.33E-1	-7.79E-2	-1.72E-1	-3.50E-1	-5.40E-2	-1.51E-1	-3.38E-1	-4.02E-2	-1.40E-1	-3.34E-1
6	-4.85E-1	-1.21E00	-1.53E00	-1.00E00	-1.41E00	-1.57E00	-8.13E-1	-1.30E00	-1.54E00	-6.91E-1	-1.25E00	-1.53E00

Table 2.43: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{7}$, and $q = 1$

k=7 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-8.53E-11	-2.39E-7	-6.80E-6	-1.82E-6	-5.18E-6	-1.75E-5	-3.22E-7	-1.50E-6	-9.46E-6	-6.14E-8	-5.98E-7	-7.41E-6
2	-3.02E-8	-1.29E-5	-1.69E-4	-2.46E-5	-8.22E-5	-2.84E-4	-6.22E-6	-3.63E-5	-2.01E-4	-1.79E-6	-2.11E-5	-1.77E-4
3	-2.19E-6	-2.48E-4	-1.86E-3	-2.41E-4	-8.29E-4	-2.57E-3	-8.50E-5	-4.78E-4	-2.07E-3	-3.40E-5	-3.36E-4	-1.91E-3
4	-7.68E-5	-2.88E-3	-1.38E-2	-2.08E-3	-6.42E-3	-1.71E-2	-9.65E-4	-4.43E-3	-1.48E-2	-4.99E-4	-3.51E-3	-1.41E-2
5	-1.77E-3	-2.48E-2	-8.19E-2	-1.65E-2	-4.14E-2	-9.35E-2	-9.62E-3	-3.25E-2	-8.55E-2	-6.11E-3	-2.81E-2	-8.28E-2
6	-3.18E-2	-1.77E-1	-4.09E-1	-1.28E-1	-2.43E-1	-4.40E-1	-9.06E-2	-2.09E-1	-4.19E-1	-6.80E-2	-1.91E-1	-4.12E-1
7	-6.10E-1	-1.43E00	-1.73E00	-1.36E00	-1.75E00	-1.82E00	-1.10E00	-1.59E00	-1.76E00	-9.27E-1	-1.50E00	-1.74E00

Table 2.44: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{8}$, and $q = 1$

k=8 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-9.64E-12	-5.02E-8	-1.88E-6	-1.25E-6	-3.06E-6	-8.44E-6	-2.09E-7	-7.40E-7	-3.54E-6	-3.64E-8	-2.31E-7	-2.28E-6
2	-3.40E-9	-2.72E-6	-4.68E-5	-1.36E-5	-3.79E-5	-1.13E-4	-3.05E-6	-1.37E-5	-6.63E-5	-7.56E-7	-6.44E-6	-5.18E-5
3	-2.44E-7	-5.21E-5	-5.14E-4	-1.07E-4	-3.18E-4	-9.17E-4	-3.23E-5	-1.53E-4	-6.42E-4	-1.09E-5	-9.15E-5	-5.48E-4
4	-8.43E-6	-6.07E-4	-3.79E-3	-7.45E-4	-2.16E-3	-5.63E-3	-2.94E-4	-1.28E-3	-4.41E-3	-1.29E-4	-8.90E-4	-3.96E-3
5	-1.89E-4	-5.20E-3	-2.21E-2	-4.85E-3	-1.25E-2	-2.89E-2	-2.41E-3	-8.65E-3	-2.44E-2	-1.31E-3	-6.75E-3	-2.27E-2
6	-3.18E-3	-3.61E-2	-1.10E-1	-3.01E-2	-6.48E-2	-1.31E-1	-1.83E-2	-5.05E-2	-1.18E-1	-1.19E-2	-4.28E-2	-1.12E-1
7	-4.54E-2	-2.23E-1	-4.83E-1	-1.92E-1	-3.24E-1	-5.33E-1	-1.37E-1	-2.75E-1	-5.01E-1	-1.04E-1	-2.48E-1	-4.88E-1
8	-7.38E-1	-1.65E00	-1.93E00	-1.76E00	-2.11E00	-2.08E00	-1.42E00	-1.89E00	-1.99E00	-1.19E00	-1.77E00	-1.95E00

Table 2.45: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k}=\mathbf{3}$, and $q = 100$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.13E-11	-1.69E-6	-1.17E-4	-2.60E-6	-1.11E-5	-1.46E-4	-3.72E-7	-4.44E-6	-1.23E-4	-5.70E-8	-2.54E-6	-1.18E-4
2	-8.24E-9	-8.15E-5	-1.81E-3	-1.36E-4	-2.44E-4	-1.95E-3	-2.44E-5	-1.36E-4	-1.84E-3	-4.62E-6	-1.00E-4	-1.81E-3
3	-5.38E-6	-3.29E-3	-1.92E-2	-1.41E-2	-8.79E-3	-2.12E-2	-3.85E-3	-5.19E-3	-1.96E-2	-1.01E-3	-3.94E-3	-1.93E-2

Table 2.46: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k}=\mathbf{4}$, and $q = 100$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.09E-12	-3.09E-7	-2.59E-5	-1.51E-6	-5.49E-6	-4.36E-5	-2.14E-7	-1.68E-6	-3.01E-5	-3.18E-8	-7.08E-7	-2.68E-5
2	-5.53E-10	-1.57E-5	-5.91E-4	-3.97E-5	-8.48E-5	-7.44E-4	-6.76E-6	-3.92E-5	-6.31E-4	-1.21E-6	-2.40E-5	-6.00E-4
3	-1.01E-7	-3.05E-4	-3.66E-3	-1.16E-3	-1.18E-3	-4.25E-3	-2.68E-4	-6.15E-4	-3.81E-3	-6.14E-5	-4.17E-4	-3.69E-3
4	-4.08E-5	-1.07E-2	-4.71E-2	-6.07E-2	-3.40E-2	-5.57E-2	-2.27E-2	-1.98E-2	-4.93E-2	-7.80E-3	-1.41E-2	-4.76E-2

Table 2.47: The poles of $r(q,\delta,\alpha,k;t)$, (1.42), of $r_{q,\delta,\alpha,k}(t)$, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k}=\mathbf{5}$, and $q = 100$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.36E-13	-6.38E-8	-6.47E-6	-1.00E-6	-3.21E-6	-1.70E-5	-1.41E-7	-8.09E-7	-9.09E-6	-2.05E-8	-2.63E-7	-7.06E-6
2	-5.88E-11	-3.37E-6	-1.61E-4	-1.75E-5	-3.90E-5	-2.71E-4	-2.86E-6	-1.49E-5	-1.92E-4	-4.87E-7	-7.36E-6	-1.69E-4
3	-6.77E-9	-5.97E-5	-1.36E-3	-2.68E-4	-3.54E-4	-1.76E-3	-5.62E-5	-1.66E-4	-1.48E-3	-1.19E-5	-1.00E-4	-1.39E-3
4	-7.18E-7	-8.70E-4	-7.06E-3	-5.21E-3	-4.17E-3	-9.24E-3	-1.52E-3	-2.12E-3	-7.69E-3	-4.28E-4	-1.36E-3	-7.21E-3
5	-2.13E-4	-2.64E-2	-9.23E-2	-1.56E-1	-8.97E-2	-1.17E-1	-7.23E-2	-5.40E-2	-9.97E-2	-3.13E-2	-3.80E-2	-9.41E-2

Table 2.48: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k}=\mathbf{6}$, and $q = 100$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-2.00E-14	-1.44E-8	-1.78E-6	-7.16E-7	-2.10E-6	-8.23E-6	-1.01E-7	-4.61E-7	-3.41E-6	-1.45E-8	-1.23E-7	-2.17E-6
2	-7.94E-12	-7.75E-7	-4.47E-5	-9.60E-6	-2.11E-5	-1.10E-4	-1.52E-6	-6.78E-6	-6.39E-5	-2.49E-7	-2.74E-6	-4.96E-5
3	-7.40E-10	-1.42E-5	-4.68E-4	-9.83E-5	-1.50E-4	-7.97E-4	-1.92E-5	-6.25E-5	-5.78E-4	-3.83E-6	-3.27E-5	-4.98E-4
4	-4.60E-8	-1.60E-4	-2.35E-3	-1.12E-3	-1.13E-3	-3.30E-3	-2.84E-4	-5.24E-4	-2.67E-3	-7.14E-5	-3.06E-4	-2.44E-3
5	-3.61E-6	-2.06E-3	-1.29E-2	-1.54E-2	-1.13E-2	-1.90E-2	-5.45E-3	-5.86E-3	-1.49E-2	-1.84E-3	-3.65E-3	-1.35E-2
6	-8.23E-4	-5.27E-2	-1.54E-1	-3.01E-1	-1.83E-1	-2.08E-1	-1.60E-1	-1.15E-1	-1.73E-1	-8.17E-2	-8.14E-2	-1.59E-1

Table 2.49: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=7}$, and $q = 100$

k=7 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-3.29E-15	-3.53E-9	-5.30E-7	-5.41E-7	-1.48E-6	-4.68E-6	-7.58E-8	-2.95E-7	-1.56E-6	-1.08E-8	-6.78E-8	-7.86E-7
2	-1.24E-12	-1.91E-7	-1.32E-5	-6.04E-6	-1.28E-5	-5.16E-5	-9.28E-7	-3.55E-6	-2.48E-5	-1.47E-7	-1.20E-6	-1.65E-5
3	-1.04E-10	-3.59E-6	-1.45E-4	-4.69E-5	-7.71E-5	-3.61E-4	-8.65E-6	-2.83E-5	-2.19E-4	-1.64E-6	-1.27E-5	-1.67E-4
4	-4.98E-9	-3.95E-5	-9.42E-4	-3.76E-4	-4.43E-4	-1.62E-3	-8.67E-5	-1.89E-4	-1.20E-3	-2.01E-5	-1.00E-4	-1.02E-3
5	-2.20E-7	-3.56E-4	-3.73E-3	-3.34E-3	-2.94E-3	-6.03E-3	-1.00E-3	-1.38E-3	-4.56E-3	-2.94E-4	-7.87E-4	-3.99E-3
6	-1.40E-5	-4.18E-3	-2.17E-2	-3.45E-2	-2.48E-2	-3.51E-2	-1.42E-2	-1.34E-2	-2.67E-2	-5.60E-3	-8.32E-3	-2.33E-2
7	-2.48E-3	-9.00E-2	-2.30E-1	-4.93E-1	-3.13E-1	-3.29E-1	-2.87E-1	-2.05E-1	-2.69E-1	-1.63E-1	-1.47E-1	-2.42E-1

Table 2.50: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=8}$, and $q = 100$

k=8 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-7.52E-16	-9.23E-10	-1.68E-7	-4.25E-7	-1.10E-6	-2.98E-6	-5.94E-8	-2.05E-7	-8.32E-7	-8.39E-9	-4.22E-8	-3.35E-7
2	3.96E-14	-4.99E-8	-4.19E-6	-4.15E-6	-8.40E-6	-2.78E-5	-6.24E-7	-2.08E-6	-1.11E-5	-9.63E-8	-6.07E-7	-6.22E-6
3	1.25E-11	-9.50E-7	-4.62E-5	-2.64E-5	-4.46E-5	-1.75E-4	-4.65E-6	-1.45E-5	-9.03E-5	-8.40E-7	-5.63E-6	-6.02E-5
4	-3.72E-9	-1.07E-5	-3.33E-4	-1.65E-4	-2.15E-4	-8.29E-4	-3.53E-5	-8.39E-5	-5.27E-4	-7.64E-6	-3.99E-5	-4.00E-4
5	3.72E-9	-8.78E-5	-1.54E-3	-1.09E-3	-1.10E-3	-2.78E-3	-2.92E-4	-4.76E-4	-2.05E-3	-7.74E-5	-2.51E-4	-1.72E-3
6	3.35E-6	-7.02E-4	-5.75E-3	-7.88E-3	-6.51E-3	-1.07E-2	-2.72E-3	-3.13E-3	-7.69E-3	-9.14E-4	-1.78E-3	-6.41E-3
7	2.46E-5	-7.55E-3	-3.36E-2	-6.42E-2	-4.66E-2	-5.90E-2	-2.97E-2	-2.61E-2	-4.40E-2	-1.32E-2	-1.65E-2	-3.73E-2
8	1.03E-2	-1.38E-1	-3.18E-1	-7.30E-1	-4.80E-1	-4.79E-1	-4.50E-1	-3.24E-1	-3.87E-1	-2.75E-1	-2.37E-1	-3.43E-1

Table 2.51: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=3}$, and $q = 200$

k=3 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-7.62E-13	-4.65E-7	-5.18E-5	-2.21E-6	-5.95E-6	-7.36E-5	-2.87E-7	-1.99E-6	-5.68E-5	-3.87E-8	-9.21E-7	-5.28E-5
2	-5.67E-10	-2.24E-5	-7.69E-4	-1.17E-4	-1.16E-4	-8.72E-4	-1.85E-5	-5.15E-5	-7.94E-4	-2.92E-6	-3.22E-5	-7.74E-4
3	-3.98E-7	-9.78E-4	-9.03E-3	-1.24E-2	-4.63E-3	-1.08E-2	-3.00E-3	-2.15E-3	-9.44E-3	-6.48E-4	-1.37E-3	-9.12E-3

Table 2.52: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=4}$, and $q = 200$

k=4 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-7.64E-14	-9.00E-8	-1.21E-5	-1.31E-6	-3.41E-6	-2.54E-5	-1.72E-7	-9.12E-7	-1.53E-5	-2.30E-8	-3.17E-7	-1.28E-5
2	-3.91E-11	-4.53E-6	-2.69E-4	-3.51E-5	-4.49E-5	-3.74E-4	-5.42E-6	-1.72E-5	-2.98E-4	-8.33E-7	-8.90E-6	-2.76E-4
3	-7.43E-9	-9.08E-5	-1.65E-3	-1.04E-3	-6.54E-4	-2.11E-3	-2.16E-4	-2.71E-4	-1.78E-3	-4.20E-5	-1.54E-4	-1.68E-3
4	-3.37E-6	-3.68E-3	-2.49E-2	-5.61E-2	-2.14E-2	-3.28E-2	-1.92E-2	-1.00E-2	-2.71E-2	-5.71E-3	-6.03E-3	-2.54E-2

Table 2.53: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=5}$, and $q = 200$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-9.97E-15	-1.97E-8	-3.15E-6	-8.77E-7	-2.20E-6	-1.12E-5	-1.16E-7	-5.07E-7	-5.18E-6	-1.55E-8	-1.41E-7	-3.63E-6
2	-4.33E-12	-1.04E-6	-7.83E-5	-1.58E-5	-2.28E-5	-1.56E-4	-2.38E-6	-7.56E-6	-1.01E-4	-3.58E-7	-3.21E-6	-8.41E-5
3	-5.11E-10	-1.83E-5	-6.27E-4	-2.43E-4	-2.06E-4	-8.91E-4	-4.67E-5	-7.88E-5	-7.11E-4	-8.63E-6	-4.03E-5	-6.48E-4
4	-5.77E-8	-2.87E-4	-3.50E-3	-4.79E-3	-2.64E-3	-5.30E-3	-1.29E-3	-1.10E-3	-4.04E-3	-3.18E-4	-5.89E-4	-3.64E-3
5	-2.05E-5	-1.05E-2	-5.40E-2	-1.48E-1	-6.45E-2	-7.77E-2	-6.47E-2	-3.27E-2	-6.15E-2	-2.53E-2	-1.96E-2	-5.60E-2

Table 2.54: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=6}$, and $q = 200$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.54E-15	-4.71E-9	-9.03E-7	-6.32E-7	-1.54E-6	-5.94E-6	-8.37E-8	-3.19E-7	-2.16E-6	-1.12E-8	-7.57E-8	-1.21E-6
2	-6.16E-13	-2.53E-7	-2.27E-5	-8.77E-6	-1.35E-5	-7.07E-5	-1.29E-6	-3.90E-6	-3.71E-5	-1.91E-7	-1.38E-6	-2.66E-5
3	-5.84E-11	-4.59E-6	-2.32E-4	-9.01E-5	-9.19E-5	-4.49E-4	-1.63E-5	-3.23E-5	-3.09E-4	-2.90E-6	-1.47E-5	-2.55E-4
4	-3.79E-9	-5.29E-5	-1.12E-3	-1.03E-3	-7.30E-4	-1.83E-3	-2.44E-4	-2.79E-4	-1.36E-3	-5.46E-5	-1.38E-4	-1.19E-3
5	-3.27E-7	-7.57E-4	-7.00E-3	-1.44E-2	-7.94E-3	-1.22E-2	-4.81E-3	-3.46E-3	-8.77E-3	-1.47E-3	-1.84E-3	-7.51E-3
6	-9.57E-5	-2.42E-2	-9.77E-2	-2.89E-1	-1.43E-1	-1.51E-1	-1.48E-1	-7.87E-2	-1.17E-1	-7.01E-2	-4.89E-2	-1.03E-1

Table 2.55: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=7}$, and $q = 200$

k=7 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-9.85E-15	-1.21E-9	-2.78E-7	-4.79E-7	-1.14E-6	-3.61E-6	-6.37E-8	-2.18E-7	-1.08E-6	-8.54E-9	-4.61E-8	-4.80E-7
2	-4.28E-12	-6.55E-8	-6.96E-6	-5.57E-6	-8.79E-6	-3.61E-5	-8.07E-7	-2.26E-6	-1.57E-5	-1.17E-7	-6.85E-7	-9.46E-6
3	-5.07E-10	-1.23E-6	-7.61E-5	-4.34E-5	-4.96E-5	-2.28E-4	-7.50E-6	-1.59E-5	-1.30E-4	-1.28E-6	-6.38E-6	-9.29E-5
4	1.65E-9	-1.35E-5	-4.68E-4	-3.49E-4	-2.92E-4	-9.06E-4	-7.56E-5	-1.04E-4	-6.39E-4	-1.57E-5	-4.78E-5	-5.26E-4
5	-5.79E-8	-1.28E-4	-1.89E-3	-3.14E-3	-2.07E-3	-3.74E-3	-8.89E-4	-8.18E-4	-2.56E-3	-2.36E-4	-3.99E-4	-2.10E-3
6	-2.05E-5	-1.71E-3	-1.27E-2	-3.28E-2	-1.88E-2	-2.45E-2	-1.29E-2	-8.74E-3	-1.72E-2	-4.68E-3	-4.73E-3	-1.41E-2
7	1.51E-3	-4.63E-2	-1.55E-1	-4.78E-1	-2.60E-1	-2.54E-1	-2.70E-1	-1.53E-1	-1.95E-1	-1.45E-1	-9.90E-2	-1.69E-1

Table 2.56: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q,δ,α,k)-BURA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=8}$, and $q = 200$

k=8 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	6.67E-15	-3.32E-10	-9.09E-8	-3.77E-7	-8.79E-7	-2.41E-6	-5.03E-8	-1.59E-7	-6.21E-7	-6.75E-9	-3.08E-8	-2.23E-7
2	2.42E-13	-1.79E-8	-2.27E-6	-3.86E-6	-6.13E-6	-2.07E-5	-5.50E-7	-1.43E-6	-7.57E-6	-7.86E-8	-3.81E-7	-3.83E-6
3	2.39E-9	-3.41E-7	-2.50E-5	-2.46E-5	-3.02E-5	-1.21E-4	-4.09E-6	-8.86E-6	-5.78E-5	-6.76E-7	-3.12E-6	-3.57E-5
4	1.00E-8	-3.82E-6	-1.78E-4	-1.54E-4	-1.45E-4	-5.07E-4	-3.11E-5	-4.87E-5	-3.12E-4	-6.13E-6	-2.04E-5	-2.26E-4
5	4.96E-8	-3.16E-5	-7.72E-4	-1.03E-3	-7.80E-4	-1.66E-3	-2.60E-4	-2.86E-4	-1.12E-3	-6.29E-5	-1.30E-4	-8.99E-4
6	5.10E-7	-2.74E-4	-3.13E-3	-7.49E-3	-4.88E-3	-7.28E-3	-2.47E-3	-2.03E-3	-4.75E-3	-7.62E-4	-9.99E-4	-3.70E-3
7	3.81E-5	-3.40E-3	-2.08E-2	-6.18E-2	-3.73E-2	-4.38E-2	-2.75E-2	-1.85E-2	-3.03E-2	-1.15E-2	-1.03E-2	-2.43E-2
8	9.99E-3	-7.77E-2	-2.25E-1	-7.11E-1	-4.13E-1	-3.87E-1	-4.29E-1	-2.57E-1	-2.97E-1	-2.52E-1	-1.72E-1	-2.52E-1

Table 2.57: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 3}$, and $q = 400$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-4.95E-14	-1.22E-7	-2.20E-5	-2.02E-6	-3.37E-6	-3.77E-5	-2.46E-7	-9.68E-7	-2.58E-5	-3.02E-8	-3.65E-7	-2.28E-5
2	-3.72E-11	-5.90E-6	-3.19E-4	-1.08E-4	-6.26E-5	-3.92E-4	-1.58E-5	-2.14E-5	-3.38E-4	-2.22E-6	-1.09E-5	-3.23E-4
3	-2.71E-8	-2.70E-4	-4.02E-3	-1.16E-2	-2.72E-3	-5.41E-3	-2.60E-3	-9.55E-4	-4.37E-3	-4.94E-4	-4.89E-4	-4.10E-3

Table 2.58: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 4}$, and $q = 400$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-5.06E-15	-2.46E-8	-5.36E-6	-1.21E-6	-2.15E-6	-1.51E-5	-1.50E-7	-5.23E-7	-7.80E-6	-1.85E-8	-1.53E-7	-5.92E-6
2	-2.61E-12	-1.24E-6	-1.17E-4	-3.29E-5	-2.62E-5	-1.86E-4	-4.78E-6	-8.07E-6	-1.38E-4	-6.66E-7	-3.47E-6	-1.22E-4
3	-5.04E-10	-2.52E-5	-7.21E-4	-9.75E-4	-4.05E-4	-1.07E-3	-1.91E-4	-1.30E-4	-8.17E-4	-3.34E-5	-5.94E-5	-7.44E-4
4	-2.44E-7	-1.12E-3	-1.23E-2	-5.37E-2	-1.45E-2	-1.90E-2	-1.75E-2	-5.33E-3	-1.42E-2	-4.71E-3	-2.57E-3	-1.27E-2

Table 2.59: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 5}$, and $q = 400$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-6.76E-16	-5.63E-9	-1.46E-6	-8.11E-7	-1.50E-6	-7.48E-6	-1.02E-7	-3.26E-7	-2.98E-6	-1.27E-8	-8.01E-8	-1.83E-6
2	-2.95E-13	-2.96E-7	-3.61E-5	-1.49E-5	-1.43E-5	-8.95E-5	-2.14E-6	-4.00E-6	-5.26E-5	-2.97E-7	-1.45E-6	-4.05E-5
3	-3.53E-11	-5.24E-6	-2.77E-4	-2.30E-4	-1.33E-4	-4.54E-4	-4.20E-5	-4.03E-5	-3.35E-4	-7.10E-6	-1.69E-5	-2.93E-4
4	-4.12E-9	-8.64E-5	-1.66E-3	-4.58E-3	-1.83E-3	-3.08E-3	-1.18E-3	-6.04E-4	-2.10E-3	-2.66E-4	-2.63E-4	-1.78E-3
5	-1.62E-6	-3.66E-3	-2.94E-2	-1.43E-1	-4.89E-2	-5.10E-2	-6.07E-2	-2.03E-2	-3.65E-2	-2.21E-2	-1.00E-2	-3.14E-2

Table 2.60: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 6}$, and $q = 400$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.08E-16	-1.42E-9	-4.34E-7	-5.86E-7	-1.12E-6	-4.33E-6	-7.42E-8	-2.23E-7	-1.39E-6	-9.35E-9	-4.82E-8	-6.73E-7
2	-4.33E-14	-7.59E-8	-1.09E-5	-8.35E-6	-8.95E-6	-4.53E-5	-1.18E-6	-2.29E-6	-2.14E-5	-1.63E-7	-7.17E-7	-1.39E-5
3	-4.14E-12	-1.37E-6	-1.09E-4	-8.60E-5	-6.14E-5	-2.47E-4	-1.49E-5	-1.76E-5	-1.60E-4	-2.45E-6	-6.77E-6	-1.25E-4
4	-2.75E-10	-1.61E-5	-5.15E-4	-9.90E-4	-5.15E-4	-1.04E-3	-2.24E-4	-1.60E-4	-6.91E-4	-4.64E-5	-6.48E-5	-5.68E-4
5	-2.51E-8	-2.51E-4	-3.60E-3	-1.40E-2	-5.95E-3	-7.87E-3	-4.48E-3	-2.14E-3	-5.08E-3	-1.28E-3	-9.42E-4	-4.06E-3
6	-8.52E-6	-9.61E-3	-5.81E-2	-2.83E-1	-1.17E-1	-1.09E-1	-1.41E-1	-5.52E-2	-7.70E-2	-6.39E-2	-2.91E-2	-6.41E-2

Table 2.61: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 7}$, and $q = 400$

k=7 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	9.05E-14	-3.83E-10	-1.38E-7	-4.45E-7	-8.62E-7	-2.79E-6	-5.67E-8	-1.62E-7	-7.58E-7	-7.20E-9	-3.20E-8	-2.93E-7
2	-1.37E-13	-2.06E-8	-3.47E-6	-5.32E-6	-6.13E-6	-2.52E-5	-7.43E-7	-1.45E-6	-9.86E-6	-1.01E-7	-3.97E-7	-5.34E-6
3	3.25E-13	-3.85E-7	-3.77E-5	-4.16E-5	-3.42E-5	-1.40E-4	-6.92E-6	-9.29E-6	-7.50E-5	-1.10E-6	-3.24E-6	-5.00E-5
4	4.90E-10	-4.22E-6	-2.21E-4	-3.36E-4	-2.10E-4	-5.11E-4	-6.99E-5	-6.17E-5	-3.32E-4	-1.36E-5	-2.36E-5	-2.60E-4
5	3.57E-9	-4.21E-5	-9.30E-4	-3.03E-3	-1.56E-3	-2.39E-3	-8.29E-4	-5.15E-4	-1.45E-3	-2.07E-4	-2.10E-4	-1.10E-3
6	7.90E-8	-6.29E-4	-7.01E-3	-3.20E-2	-1.50E-2	-1.72E-2	-1.22E-2	-5.95E-3	-1.09E-2	-4.21E-3	-2.73E-3	-8.34E-3
7	2.77E-4	-2.10E-2	-9.93E-2	-4.70E-1	-2.23E-1	-1.96E-1	-2.61E-1	-1.17E-1	-1.39E-1	-1.36E-1	-6.63E-2	-1.14E-1

Table 2.62: The poles of $r_{q,\delta,\alpha,k}(t)$, (1.42), of (q, δ, α, k)-BURA approximation, i.e. the best uniform rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 8}$, and $q = 400$

k=8 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	1.55E-13	-1.10E-10	-4.66E-8	-3.51E-7	-6.88E-7	-1.94E-6	-4.49E-8	-1.23E-7	-4.65E-7	-5.73E-9	-2.27E-8	-1.49E-7
2	2.42E-11	-5.92E-9	-1.16E-6	-3.70E-6	-4.47E-6	-1.53E-5	-5.10E-7	-9.81E-7	-5.13E-6	-6.91E-8	-2.42E-7	-2.34E-6
3	-1.05E-10	-1.12E-7	-1.28E-5	-2.37E-5	-2.14E-5	-8.14E-5	-3.81E-6	-5.51E-6	-3.64E-5	-5.92E-7	-1.74E-6	-2.07E-5
4	3.87E-10	-1.25E-6	-8.93E-5	-1.49E-4	-1.06E-4	-3.00E-4	-2.90E-5	-2.97E-5	-1.77E-4	-5.37E-6	-1.07E-5	-1.23E-4
5	-3.93E-9	-1.05E-5	-3.71E-4	-9.94E-4	-5.94E-4	-1.03E-3	-2.44E-4	-1.83E-4	-6.18E-4	-5.56E-5	-7.02E-5	-4.62E-4
6	-1.59E-7	-9.78E-5	-1.65E-3	-7.29E-3	-3.88E-3	-5.05E-3	-2.33E-3	-1.38E-3	-2.95E-3	-6.84E-4	-5.79E-4	-2.13E-3
7	-1.55E-5	-1.37E-3	-1.23E-2	-6.06E-2	-3.12E-2	-3.26E-2	-2.64E-2	-1.36E-2	-2.07E-2	-1.06E-2	-6.57E-3	-1.55E-2
8	-3.37E-3	-3.93E-2	-1.52E-1	-7.01E-1	-3.66E-1	-3.13E-1	-4.17E-1	-2.08E-1	-2.24E-1	-2.39E-1	-1.25E-1	-1.81E-1

2.4 Tables type (d) for the coefficients of partial fractions representation of BURA

Table 2.63: The coefficients c_j, $j = 0, \dots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 3}$, and $q = 0$

k=3 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	1.23E00	2.28E00	6.15E00	1.29E00	2.31E00	6.17E00	1.27E00	2.29E00	6.16E00	1.25E00	2.28E00	6.15E00
1	-1.40E-6	-4.18E-5	-9.88E-5	-1.12E-5	-5.15E-5	-1.01E-4	-5.10E-6	-4.48E-5	-9.92E-5	-3.02E-6	-4.28E-5	-9.88E-5
2	-1.37E-3	-1.20E-2	-2.39E-2	-2.99E-3	-1.29E-2	-2.41E-2	-2.19E-3	-1.23E-2	-2.39E-2	-1.80E-3	-1.21E-2	-2.39E-2
3	-3.44E-1	-3.52E00	-3.64E01	-4.60E-1	-3.65E00	-3.66E01	-4.07E-1	-3.56E00	-3.65E01	-3.79E-1	-3.54E00	-3.64E01

Table 2.64: The coefficients c_j, $j = 0, \dots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 4}$, and $q = 0$

k=4 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	1.32E00	2.64E00	7.74E00	1.42E00	2.70E00	7.78E00	1.38E00	2.66E00	7.75E00	1.36E00	2.65E00	7.74E00
1	-2.61E-8	-1.40E-6	-3.71E-6	-1.33E-6	-2.62E-6	-4.07E-6	-3.77E-7	-1.77E-6	-3.78E-6	-1.44E-7	-1.52E-6	-3.72E-6
2	-2.53E-5	-3.73E-4	-7.20E-4	-1.63E-4	-4.87E-4	-7.49E-4	-8.52E-5	-4.11E-4	-7.26E-4	-5.35E-5	-3.85E-4	-7.21E-4
3	-4.22E-3	-2.90E-2	-5.64E-2	-1.00E-2	-3.29E-2	-5.75E-2	-7.33E-3	-3.03E-2	-5.66E-2	-5.90E-3	-2.94E-2	-5.64E-2
4	-5.29E-1	-5.63E00	-6.29E01	-7.87E-1	-6.05E00	-6.38E01	-6.76E-1	-5.77E00	-6.31E01	-6.12E-1	-5.68E00	-6.30E01

Table 2.65: The coefficients c_j, $j = 0, \dots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 5}$, and $q = 0$

k=5 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	1.40E00	2.96E00	9.24E00	1.54E00	3.08E00	9.36E00	1.49E00	3.01E00	9.27E00	1.45E00	2.98E00	9.25E00
1	-7.46E-10	-6.83E-8	-2.03E-7	-3.25E-7	-2.76E-7	-2.69E-7	-6.50E-8	-1.24E-7	-2.17E-7	-1.65E-8	-8.49E-8	-2.06E-7
2	-7.24E-7	-1.80E-5	-3.80E-5	-2.02E-5	-3.56E-5	-4.34E-5	-7.52E-6	-2.39E-5	-3.92E-5	-3.43E-6	-1.99E-5	-3.83E-5
3	-1.19E-4	-1.24E-3	-2.18E-3	-7.67E-4	-1.78E-3	-2.34E-3	-4.29E-4	-1.43E-3	-2.21E-3	-2.74E-4	-1.30E-3	-2.18E-3
4	-8.58E-3	-5.21E-2	-1.01E-1	-2.23E-2	-6.31E-2	-1.06E-1	-1.64E-2	-5.63E-2	-1.02E-1	-1.30E-2	-5.35E-2	-1.02E-1
5	-7.29E-1	-8.07E00	-9.60E01	-1.21E00	-9.17E00	-9.89E01	-1.02E00	-8.49E00	-9.66E01	-8.99E-1	-8.21E00	-9.61E01

Table 2.66: The coefficients c_j, $j = 0, \dots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 6}$, and $q = 0$

k=6 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	1.47E00	3.25E00	1.07E01	1.65E00	3.46E00	1.10E01	1.59E00	3.35E00	1.08E01	1.54E00	3.29E00	1.07E01
1	-2.93E-11	-4.36E-9	-1.45E-8	-1.22E-7	-5.26E-8	-2.84E-8	-1.91E-8	-1.52E-8	-1.74E-8	-3.58E-9	-7.30E-9	-1.51E-8
2	-2.84E-8	-1.15E-6	-2.70E-6	-4.32E-6	-4.42E-6	-3.80E-6	-1.22E-6	-2.18E-6	-2.96E-6	-4.11E-7	-1.49E-6	-2.75E-6
3	-4.65E-6	-7.74E-5	-1.45E-4	-1.09E-4	-1.72E-4	-1.77E-4	-4.67E-5	-1.12E-4	-1.53E-4	-2.32E-5	-8.97E-5	-1.47E-4
4	-3.26E-4	-2.74E-3	-4.59E-3	-2.17E-3	-4.36E-3	-5.16E-3	-1.28E-3	-3.40E-3	-4.74E-3	-8.35E-4	-2.98E-3	-4.63E-3
5	-1.42E-2	-8.01E-2	-1.58E-1	-3.98E-2	-1.05E-1	-1.70E-1	-2.95E-2	-9.06E-2	-1.61E-1	-2.33E-2	-8.41E-2	-1.59E-1
6	-9.41E-1	-1.08E01	-1.35E02	-1.74E00	-1.31E01	-1.43E02	-1.44E00	-1.18E01	-1.37E02	-1.24E00	-1.12E01	-1.36E02

Table 2.67: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 7}$, and $q = 0$

k=7 $/\delta$	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j $/\alpha$	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	1.53E00	3.52E00	1.21E01	1.75E00	3.84E00	1.26E01	1.68E00	3.68E00	1.22E01	1.63E00	3.59E00	1.21E01
1	-1.47E-12	-3.43E-10	-1.27E-9	-5.85E-8	-1.54E-8	-4.69E-9	-7.84E-9	-3.03E-9	-1.96E-9	-1.19E-9	-9.71E-10	-1.41E-9
2	-1.43E-9	-9.02E-8	-2.35E-7	-1.34E-6	-8.43E-7	-4.81E-7	-3.02E-7	-3.03E-7	-2.96E-7	-7.95E-8	-1.57E-7	-2.49E-7
3	-2.33E-7	-6.06E-6	-1.25E-5	-2.38E-5	-2.52E-5	-1.95E-5	-8.08E-6	-1.29E-5	-1.44E-5	-3.17E-6	-8.52E-6	-1.30E-5
4	-1.63E-5	-2.09E-4	-3.64E-4	-3.53E-4	-5.18E-4	-4.81E-4	-1.67E-4	-3.35E-4	-3.98E-4	-8.82E-5	-2.58E-4	-3.72E-4
5	-6.74E-4	-4.90E-3	-8.00E-3	-4.62E-3	-8.60E-3	-9.53E-3	-2.84E-3	-6.55E-3	-8.45E-3	-1.90E-3	-5.57E-3	-8.11E-3
6	-2.08E-2	-1.13E-1	-2.26E-1	-6.27E-2	-1.60E-1	-2.54E-1	-4.67E-2	-1.34E-1	-2.35E-1	-3.69E-2	-1.22E-1	-2.28E-1
7	-1.16E00	-1.38E01	-1.81E02	-2.38E00	-1.81E01	-1.99E02	-1.94E00	-1.58E01	-1.86E02	-1.66E00	-1.47E01	-1.82E02

Table 2.68: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 8}$, and $q = 0$

k=8 $/\delta$	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j $/\alpha$	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	1.58E00	3.77E00	1.34E01	1.84E00	4.22E00	1.43E01	1.77E00	4.01E00	1.37E01	1.71E00	3.89E00	1.35E01
1	-8.95E-14	-3.18E-11	-1.30E-10	-3.30E-8	-6.06E-9	-1.14E-9	-3.96E-9	-8.83E-10	-3.11E-10	-5.18E-10	-1.95E-10	-1.68E-10
2	-8.69E-11	-8.38E-9	-2.42E-8	-5.40E-7	-2.25E-7	-8.52E-8	-1.02E-7	-6.01E-8	-3.91E-8	-2.19E-8	-2.33E-8	-2.77E-8
3	-1.42E-8	-5.63E-7	-1.28E-6	-7.11E-6	-5.16E-6	-2.91E-6	-1.98E-6	-2.05E-6	-1.74E-6	-6.28E-7	-1.09E-6	-1.40E-6
4	-9.92E-7	-1.93E-5	-3.66E-5	-8.19E-5	-8.73E-5	-6.31E-5	-3.16E-5	-4.60E-5	-4.48E-5	-1.37E-5	-2.98E-5	-3.87E-5
5	-4.09E-5	-4.35E-4	-7.23E-4	-8.49E-4	-1.19E-3	-1.04E-3	-4.31E-4	-7.72E-4	-8.25E-4	-2.40E-4	-5.79E-4	-7.50E-4
6	-1.18E-3	-7.70E-3	-1.24E-2	-8.27E-3	-1.48E-2	-1.57E-2	-5.27E-3	-1.11E-2	-1.35E-2	-3.59E-3	-9.23E-3	-1.27E-2
7	-2.82E-2	-1.49E-1	-3.05E-1	-9.07E-2	-2.28E-1	-3.61E-1	-6.79E-2	-1.88E-1	-3.24E-1	-5.37E-2	-1.67E-1	-3.10E-1
8	-1.40E00	-1.71E01	-2.32E02	-3.14E00	-2.41E01	-2.68E02	-2.53E00	-2.06E01	-2.45E02	-2.13E00	-1.87E01	-2.36E02

Table 2.69: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 3}$, and $q = 1$

k=3 $/\delta$	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j $/\alpha$	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	5.28E-1	6.68E-1	8.42E-1	5.46E-1	6.71E-1	8.42E-1	5.39E-1	6.69E-1	8.42E-1	5.34E-1	6.68E-1	8.42E-1
1	-2.47E-7	-1.71E-5	-5.34E-5	-3.98E-6	-2.26E-5	-5.51E-5	-1.50E-6	-1.88E-5	-5.38E-5	-7.47E-7	-1.76E-5	-5.35E-5
2	-2.39E-4	-4.34E-3	-1.36E-2	-7.37E-4	-4.80E-3	-1.37E-2	-4.83E-4	-4.49E-3	-1.36E-2	-3.64E-4	-4.39E-3	-1.36E-2
3	-4.18E-2	-2.57E-1	-6.33E-1	-6.16E-2	-2.64E-1	-6.34E-1	-5.30E-2	-2.59E-1	-6.33E-1	-4.81E-2	-2.57E-1	-6.33E-1

Table 2.70: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 4}$, and $q = 1$

k=4 $/\delta$	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j $/\alpha$	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	5.49E-1	7.02E-1	8.72E-1	5.73E-1	7.09E-1	8.73E-1	5.64E-1	7.05E-1	8.72E-1	5.58E-1	7.03E-1	8.72E-1
1	-5.56E-9	-6.39E-7	-2.12E-6	-6.63E-7	-1.40E-6	-2.39E-6	-1.63E-7	-8.61E-7	-2.17E-6	-5.23E-8	-7.08E-7	-2.13E-6
2	-5.37E-6	-1.67E-4	-4.34E-4	-5.84E-5	-2.36E-4	-4.57E-4	-2.70E-5	-1.90E-4	-4.39E-4	-1.51E-5	-1.75E-4	-4.35E-4
3	-8.43E-4	-1.07E-2	-3.38E-2	-2.66E-3	-1.25E-2	-3.47E-2	-1.81E-3	-1.14E-2	-3.40E-2	-1.36E-3	-1.10E-2	-3.39E-2
4	-6.48E-2	-3.41E-1	-7.29E-1	-1.03E-1	-3.60E-1	-7.32E-1	-8.72E-2	-3.48E-1	-7.30E-1	-7.78E-2	-3.44E-1	-7.29E-1

Table 2.71: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{5}$, and $q = 1$

k=5 $/\delta$	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j $/\alpha$	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	5.64E-1	7.28E-1	8.92E-1	5.94E-1	7.38E-1	8.93E-1	5.84E-1	7.32E-1	8.92E-1	5.77E-1	7.29E-1	8.92E-1
1	-1.83E-10	-3.36E-8	-1.22E-7	-1.99E-7	-1.73E-7	-1.71E-7	-3.61E-8	-6.97E-8	-1.32E-7	-8.04E-9	-4.42E-8	-1.24E-7
2	-1.77E-7	-8.83E-6	-2.33E-5	-9.34E-6	-2.01E-5	-2.74E-5	-3.15E-6	-1.26E-5	-2.42E-5	-1.28E-6	-1.00E-5	-2.35E-5
3	-2.83E-5	-5.87E-4	-1.43E-3	-2.78E-4	-9.16E-4	-1.56E-3	-1.44E-4	-7.08E-4	-1.46E-3	-8.48E-5	-6.29E-4	-1.43E-3
4	-1.85E-3	-1.91E-2	-6.17E-2	-6.03E-3	-2.37E-2	-6.47E-2	-4.25E-3	-2.09E-2	-6.24E-2	-3.22E-3	-1.98E-2	-6.18E-2
5	-8.80E-2	-4.18E-1	-7.94E-1	-1.50E-1	-4.54E-1	-7.99E-1	-1.26E-1	-4.33E-1	-7.95E-1	-1.12E-1	-4.23E-1	-7.95E-1

Table 2.72: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{6}$, and $q = 1$

k=6 $/\delta$	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j $/\alpha$	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	5.77E-1	7.47E-1	9.06E-1	6.12E-1	7.62E-1	9.08E-1	6.02E-1	7.54E-1	9.06E-1	5.94E-1	7.50E-1	9.06E-1
1	-8.02E-12	-2.27E-9	-9.03E-9	-8.37E-8	-3.74E-8	-1.96E-8	-1.24E-8	-9.72E-9	-1.13E-8	-2.13E-9	-4.23E-9	-9.49E-9
2	-7.77E-9	-5.97E-7	-1.70E-6	-2.39E-6	-2.81E-6	-2.53E-6	-6.22E-7	-1.28E-6	-1.90E-6	-1.91E-7	-8.20E-7	-1.74E-6
3	-1.25E-6	-4.00E-5	-9.45E-5	-4.91E-5	-1.02E-4	-1.20E-4	-1.97E-5	-6.28E-5	-1.01E-4	-9.01E-6	-4.81E-5	-9.59E-5
4	-8.38E-5	-1.35E-3	-3.22E-3	-7.88E-4	-2.32E-3	-3.73E-3	-4.42E-4	-1.75E-3	-3.35E-3	-2.72E-4	-1.50E-3	-3.25E-3
5	-3.20E-3	-2.87E-2	-9.47E-2	-1.07E-2	-3.79E-2	-1.02E-1	-7.75E-3	-3.28E-2	-9.67E-2	-5.96E-3	-3.03E-2	-9.51E-2
6	-1.11E-1	-4.89E-1	-8.38E-1	-2.03E-1	-5.47E-1	-8.46E-1	-1.70E-1	-5.15E-1	-8.40E-1	-1.49E-1	-4.99E-1	-8.39E-1

Table 2.73: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{7}$, and $q = 1$

k=7 $/\delta$	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j $/\alpha$	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	5.88E-1	7.63E-1	9.16E-1	6.27E-1	7.81E-1	9.20E-1	6.16E-1	7.73E-1	9.17E-1	6.08E-1	7.68E-1	9.16E-1
1	-4.39E-13	-1.86E-10	-8.16E-10	-4.34E-8	-1.19E-8	-3.50E-9	-5.62E-9	-2.16E-9	-1.35E-9	-8.03E-10	-6.26E-10	-9.28E-10
2	-4.25E-10	-4.91E-8	-1.52E-7	-8.38E-7	-5.89E-7	-3.41E-7	-1.78E-7	-1.96E-7	-1.99E-7	-4.34E-8	-9.45E-8	-1.63E-7
3	-6.90E-8	-3.30E-6	-8.21E-6	-1.25E-5	-1.64E-5	-1.36E-5	-4.03E-6	-7.89E-6	-9.69E-6	-1.47E-6	-4.95E-6	-8.56E-6
4	-4.70E-6	-1.12E-4	-2.49E-4	-1.56E-4	-3.17E-4	-3.46E-4	-7.04E-5	-1.96E-4	-2.77E-4	-3.51E-5	-1.45E-4	-2.56E-4
5	-1.83E-4	-2.46E-3	-5.93E-3	-1.66E-3	-4.63E-3	-7.35E-3	-9.92E-4	-3.45E-3	-6.36E-3	-6.37E-4	-2.87E-3	-6.03E-3
6	-4.82E-3	-3.91E-2	-1.31E-1	-1.64E-2	-5.45E-2	-1.46E-1	-1.22E-2	-4.66E-2	-1.35E-1	-9.49E-3	-4.23E-2	-1.32E-1
7	-1.34E-1	-5.54E-1	-8.68E-1	-2.62E-1	-6.39E-1	-8.77E-1	-2.19E-1	-5.96E-1	-8.71E-1	-1.90E-1	-5.73E-1	-8.68E-1

Table 2.74: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{8}$, and $q = 1$

k=8 $/\delta$	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j $/\alpha$	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	5.97E-1	7.76E-1	9.24E-1	6.41E-1	7.98E-1	9.29E-1	6.29E-1	7.89E-1	9.26E-1	6.21E-1	7.83E-1	9.25E-1
1	-2.87E-14	-1.80E-11	-8.62E-11	-2.58E-8	-4.97E-9	-9.02E-10	-3.03E-9	-6.80E-10	-2.29E-10	-3.81E-10	-1.38E-10	-1.16E-10
2	-2.79E-11	-4.73E-9	-1.60E-8	-3.67E-7	-1.68E-7	-6.38E-8	-6.66E-8	-4.21E-8	-2.76E-8	-1.35E-8	-1.52E-8	-1.88E-8
3	-4.54E-9	-3.18E-7	-8.56E-7	-4.20E-6	-3.62E-6	-2.12E-6	-1.12E-6	-1.36E-6	-1.21E-6	-3.34E-7	-6.78E-7	-9.47E-7
4	-3.12E-7	-1.08E-5	-2.48E-5	-4.18E-5	-5.81E-5	-4.60E-5	-1.55E-5	-2.91E-5	-3.14E-5	-6.36E-6	-1.80E-5	-2.66E-5
5	-1.24E-5	-2.41E-4	-5.16E-4	-3.68E-4	-7.45E-4	-7.88E-4	-1.82E-4	-4.65E-4	-6.06E-4	-9.69E-5	-3.37E-4	-5.41E-4
6	-3.32E-4	-3.91E-3	-9.62E-3	-2.92E-3	-7.92E-3	-1.28E-2	-1.83E-3	-5.89E-3	-1.07E-2	-1.22E-3	-4.82E-3	-9.92E-3
7	-6.64E-3	-4.98E-2	-1.67E-1	-2.31E-2	-7.30E-2	-1.93E-1	-1.74E-2	-6.20E-2	-1.77E-1	-1.37E-2	-5.57E-2	-1.70E-1
8	-1.57E-1	-6.14E-1	-8.87E-1	-3.26E-1	-7.30E-1	-8.96E-1	-2.71E-1	-6.77E-1	-8.91E-1	-2.34E-1	-6.45E-1	-8.88E-1

Table 2.75: The coefficients c_j, $j = 0, \dots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{3}$, and $q = 100$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	9.55E-3	9.81E-3	9.93E-3	9.89E-3	9.86E-3	9.93E-3	9.86E-3	9.84E-3	9.93E-3	9.83E-3	9.82E-3	9.93E-3
1	-2.87E-14	-3.41E-9	-1.37E-7	-4.26E-9	-3.69E-8	-1.96E-7	-8.28E-10	-1.16E-8	-1.50E-7	-1.56E-10	-5.71E-9	-1.40E-7
2	-3.27E-11	-4.27E-7	-1.22E-5	-1.03E-7	-1.07E-6	-1.32E-5	-2.83E-8	-6.74E-7	-1.24E-5	-7.76E-9	-5.15E-7	-1.22E-5
3	-1.44E-8	-8.08E-6	-3.78E-5	-4.72E-6	-1.30E-5	-3.81E-5	-1.96E-6	-1.01E-5	-3.79E-5	-7.62E-7	-8.84E-6	-3.78E-5

Table 2.76: The coefficients c_j, $j = 0, \dots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{4}$, and $q = 100$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	9.70E-3	9.87E-3	9.95E-3	9.91E-3	9.91E-3	9.96E-3	9.89E-3	9.90E-3	9.95E-3	9.88E-3	9.88E-3	9.95E-3
1	-1.69E-15	-2.73E-10	-8.94E-9	-2.41E-9	-1.23E-8	-2.03E-8	-4.49E-10	-2.53E-9	-1.13E-8	-7.88E-11	-8.06E-10	-9.43E-9
2	-1.69E-12	-5.47E-8	-1.95E-6	-3.02E-8	-3.38E-7	-2.84E-6	-7.83E-9	-1.55E-7	-2.17E-6	-1.98E-9	-8.97E-8	-2.00E-6
3	-3.22E-10	-1.26E-6	-1.99E-5	-4.73E-7	-2.91E-6	-2.09E-5	-1.68E-7	-2.00E-6	-2.02E-5	-5.68E-8	-1.56E-6	-2.00E-5
4	-7.00E-8	-1.42E-5	-4.02E-5	-1.19E-5	-2.33E-5	-4.07E-5	-6.43E-6	-1.86E-5	-4.04E-5	-3.18E-6	-1.61E-5	-4.03E-5

Table 2.77: The coefficients c_j, $j = 0, \dots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{5}$, and $q = 100$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	9.78E-3	9.90E-3	9.97E-3	9.92E-3	9.93E-3	9.97E-3	9.91E-3	9.92E-3	9.97E-3	9.90E-3	9.91E-3	9.97E-3
1	-1.32E-16	-2.57E-11	-7.63E-10	-1.57E-9	-5.18E-9	-3.47E-9	-2.83E-10	-7.79E-10	-1.29E-9	-4.71E-11	-1.69E-10	-8.73E-10
2	-1.28E-13	-6.22E-9	-1.68E-7	-1.31E-8	-1.25E-7	-4.02E-7	-3.22E-9	-4.09E-8	-2.25E-7	-7.63E-10	-1.70E-8	-1.81E-7
3	-2.02E-11	-2.40E-7	-7.01E-6	-1.19E-7	-1.06E-6	-9.55E-6	-3.86E-8	-6.05E-7	-7.82E-6	-1.20E-8	-3.93E-7	-7.21E-6
4	-1.71E-9	-2.46E-6	-2.26E-5	-1.30E-6	-5.44E-6	-2.28E-5	-5.69E-7	-3.95E-6	-2.27E-5	-2.35E-7	-3.14E-6	-2.26E-5
5	-2.43E-7	-2.11E-5	-4.22E-5	-2.08E-5	-3.39E-5	-4.29E-5	-1.32E-5	-2.80E-5	-4.24E-5	-7.90E-6	-2.44E-5	-4.23E-5

Table 2.78: The coefficients c_j, $j = 0, \dots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{6}$, and $q = 100$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	9.83E-3	9.92E-3	9.97E-3	9.93E-3	9.95E-3	9.98E-3	9.92E-3	9.94E-3	9.98E-3	9.91E-3	9.93E-3	9.97E-3
1	-1.22E-17	-2.77E-12	-7.88E-11	-1.12E-9	-2.59E-9	-8.81E-10	-1.95E-10	-3.09E-10	-2.16E-10	-3.12E-11	-4.93E-11	-1.07E-10
2	-1.18E-14	-7.12E-10	-1.60E-8	-7.04E-9	-5.23E-8	-6.97E-8	-1.65E-9	-1.26E-8	-2.86E-8	-3.68E-10	-3.83E-9	-1.90E-8
3	-1.82E-12	-3.85E-8	-1.03E-6	-4.55E-8	-4.54E-7	-2.55E-6	-1.37E-8	-2.03E-7	-1.48E-6	-3.99E-9	-1.02E-7	-1.14E-6
4	-1.20E-10	-5.87E-7	-1.25E-5	-3.20E-7	-2.10E-6	-1.52E-5	-1.24E-7	-1.36E-6	-1.37E-5	-4.60E-8	-9.56E-7	-1.29E-5
5	-6.37E-9	-3.88E-6	-2.28E-5	-2.59E-6	-8.36E-6	-2.28E-5	-1.34E-6	-6.33E-6	-2.28E-5	-6.49E-7	-5.12E-6	-2.28E-5
6	-6.57E-7	-2.78E-5	-4.38E-5	-3.03E-5	-4.36E-5	-4.47E-5	-2.11E-5	-3.71E-5	-4.41E-5	-1.42E-5	-3.27E-5	-4.39E-5

Table 2.79: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{7}$, and $q = 100$

k=7 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	9.85E-3	9.93E-3	9.98E-3	9.93E-3	9.96E-3	9.98E-3	9.93E-3	9.95E-3	9.98E-3	9.92E-3	9.94E-3	9.98E-3
1	-1.30E-18	-3.35E-13	-9.40E-12	-8.42E-10	-1.46E-9	-3.00E-10	-1.44E-10	-1.47E-10	-5.06E-11	-2.21E-11	-1.86E-11	-1.73E-11
2	-1.26E-15	-8.76E-11	-1.82E-9	-4.33E-9	-2.45E-8	-1.65E-8	-9.72E-10	-4.58E-9	-4.96E-9	-2.06E-10	-1.06E-9	-2.57E-9
3	-1.95E-13	-5.47E-9	-1.12E-7	-2.20E-8	-2.11E-7	-5.33E-7	-6.24E-9	-7.26E-8	-2.24E-7	-1.70E-9	-2.79E-8	-1.42E-7
4	-1.22E-11	-1.26E-7	-3.34E-6	-1.17E-7	-9.83E-7	-7.27E-6	-4.14E-8	-5.43E-7	-4.86E-6	-1.41E-8	-3.18E-7	-3.81E-6
5	-4.78E-10	-1.06E-6	-1.59E-5	-6.64E-7	-3.34E-6	-1.70E-5	-2.98E-7	-2.33E-6	-1.66E-5	-1.27E-7	-1.72E-6	-1.62E-5
6	-1.85E-8	-5.44E-6	-2.28E-5	-4.24E-6	-1.14E-5	-2.29E-5	-2.47E-6	-8.95E-6	-2.28E-5	-1.36E-6	-7.36E-6	-2.28E-5
7	-1.45E-6	-3.39E-5	-4.50E-5	-4.03E-5	-5.23E-5	-4.61E-5	-2.95E-5	-4.54E-5	-4.55E-5	-2.13E-5	-4.05E-5	-4.51E-5

Table 2.80: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{8}$, and $q = 100$

k=8 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	5.37E-3	9.94E-3	9.98E-3	9.94E-3	9.96E-3	9.99E-3	9.93E-3	9.96E-3	9.98E-3	9.93E-3	9.95E-3	9.98E-3
1	1.60E-19	-4.48E-14	-1.26E-12	-6.62E-10	-9.01E-10	-1.27E-10	-1.10E-10	-8.03E-11	-1.57E-11	-1.65E-11	-8.49E-12	-3.67E-12
2	4.31E-17	-1.18E-11	-2.38E-10	-2.93E-9	-1.27E-8	-5.02E-9	-6.31E-10	-1.94E-9	-1.13E-9	-1.27E-10	-3.59E-10	-4.46E-10
3	2.60E-20	-7.72E-10	-1.36E-8	-1.24E-8	-1.05E-7	-1.27E-7	-3.33E-9	-2.80E-8	-4.13E-8	-8.59E-10	-8.35E-9	-2.11E-8
4	3.54E-13	-2.25E-8	-4.64E-7	-5.36E-8	-5.04E-7	-2.28E-6	-1.77E-8	-2.28E-7	-1.04E-6	-5.65E-9	-1.07E-7	-6.37E-7
5	-1.36E-26	-2.83E-7	-6.85E-6	-2.41E-7	-1.64E-6	-1.18E-5	-9.79E-8	-1.03E-6	-9.36E-6	-3.81E-8	-6.71E-7	-7.82E-6
6	3.00E-25	-1.62E-6	-1.70E-5	-1.15E-6	-4.71E-6	-1.68E-5	-5.80E-7	-3.44E-6	-1.70E-5	-2.79E-7	-2.64E-6	-1.71E-5
7	8.45E-25	-7.06E-6	-2.29E-5	-6.11E-6	-1.44E-5	-2.32E-5	-3.88E-6	-1.16E-5	-2.30E-5	-2.36E-6	-9.73E-6	-2.29E-5
8	1.68E-23	-3.95E-5	-4.60E-5	-5.07E-5	-6.05E-5	-4.75E-5	-3.82E-5	-5.29E-5	-4.67E-5	-2.88E-5	-4.76E-5	-4.63E-5

Table 2.81: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{3}$, and $q = 200$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.80E-3	4.92E-3	4.97E-3	4.97E-3	4.96E-3	4.98E-3	4.96E-3	4.94E-3	4.97E-3	4.95E-3	4.93E-3	4.97E-3
1	-9.85E-16	-4.89E-10	-3.33E-8	-1.13E-9	-1.19E-8	-5.88E-8	-2.21E-10	-3.09E-9	-3.86E-8	-4.13E-11	-1.17E-9	-3.44E-8
2	-1.13E-12	-5.93E-8	-2.64E-6	-2.45E-8	-2.17E-7	-2.96E-6	-6.31E-9	-1.21E-7	-2.72E-6	-1.55E-9	-8.20E-8	-2.66E-6
3	-5.28E-10	-1.15E-6	-7.97E-6	-1.11E-6	-2.46E-6	-8.10E-6	-4.24E-7	-1.70E-6	-8.00E-6	-1.43E-7	-1.36E-6	-7.97E-6

Table 2.82: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{4}$, and $q = 200$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.87E-3	4.95E-3	4.98E-3	4.98E-3	4.97E-3	4.99E-3	4.97E-3	4.97E-3	4.99E-3	4.97E-3	4.96E-3	4.98E-3
1	-6.07E-17	-4.27E-11	-2.37E-9	-6.69E-10	-5.16E-9	-7.72E-9	-1.29E-10	-9.16E-10	-3.44E-9	-2.33E-11	-2.24E-10	-2.59E-9
2	-6.08E-14	-8.32E-9	-4.98E-7	-7.50E-9	-8.46E-8	-8.41E-7	-1.88E-9	-3.57E-8	-5.89E-7	-4.47E-10	-1.79E-8	-5.19E-7
3	-1.18E-11	-1.84E-7	-4.26E-6	-1.13E-7	-5.86E-7	-4.52E-6	-3.77E-8	-3.65E-7	-4.35E-6	-1.15E-8	-2.60E-7	-4.28E-6
4	-2.82E-9	-2.20E-6	-8.83E-6	-2.87E-6	-4.85E-6	-9.10E-6	-1.47E-6	-3.50E-6	-8.91E-6	-6.58E-7	-2.78E-6	-8.85E-6

Table 2.83: The coefficients c_j, $j = 0, \dots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{5}$, and $q = 200$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.91E-3	4.97E-3	4.99E-3	4.98E-3	4.98E-3	4.99E-3	4.98E-3	4.98E-3	4.99E-3	4.97E-3	4.97E-3	4.99E-3
1	-4.98E-18	-4.40E-12	-2.18E-10	-4.51E-10	-2.62E-9	-1.62E-9	-8.56E-11	-3.54E-10	-4.71E-10	-1.51E-11	-6.15E-11	-2.71E-10
2	-4.84E-15	-1.05E-9	-4.87E-8	-3.37E-9	-3.91E-8	-1.58E-7	-8.19E-10	-1.22E-8	-7.53E-8	-1.89E-10	-4.42E-9	-5.49E-8
3	-7.72E-13	-3.79E-8	-1.73E-6	-2.93E-8	-2.37E-7	-2.48E-6	-9.09E-9	-1.29E-7	-2.00E-6	-2.64E-9	-7.71E-8	-1.80E-6
4	-6.77E-11	-3.77E-7	-4.76E-6	-3.15E-7	-1.14E-6	-4.84E-6	-1.32E-7	-7.56E-7	-4.80E-6	-4.97E-8	-5.55E-7	-4.77E-6
5	-1.12E-8	-3.57E-6	-9.60E-6	-5.08E-6	-7.52E-6	-9.97E-6	-3.13E-6	-5.76E-6	-9.73E-6	-1.75E-6	-4.67E-6	-9.64E-6

Table 2.84: The coefficients c_j, $j = 0, \dots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{6}$, and $q = 200$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.94E-3	4.98E-3	4.99E-3	4.98E-3	4.99E-3	4.99E-3	4.98E-3	4.98E-3	4.99E-3	4.98E-3	4.98E-3	4.99E-3
1	-4.96E-19	-5.16E-13	-2.41E-11	-3.29E-10	-1.49E-9	-4.82E-10	-6.17E-11	-1.65E-10	-9.42E-11	-1.06E-11	-2.22E-11	-3.80E-11
2	-4.80E-16	-1.32E-10	-4.95E-9	-1.86E-9	-1.99E-8	-3.29E-8	-4.40E-10	-4.74E-9	-1.11E-8	-9.84E-11	-1.26E-9	-6.42E-9
3	-7.40E-14	-6.82E-9	-3.12E-7	-1.14E-8	-1.17E-7	-9.23E-7	-3.36E-9	-5.26E-8	-5.07E-7	-9.34E-10	-2.50E-8	-3.65E-7
4	-4.95E-12	-9.57E-8	-2.91E-6	-7.88E-8	-4.60E-7	-3.48E-6	-2.94E-8	-2.81E-7	-3.20E-6	-1.02E-8	-1.86E-7	-3.01E-6
5	-2.79E-10	-6.30E-7	-4.88E-6	-6.35E-7	-1.83E-6	-4.98E-6	-3.16E-7	-1.28E-6	-4.92E-6	-1.43E-7	-9.64E-7	-4.89E-6
6	-3.55E-8	-5.10E-6	-1.02E-5	-7.47E-6	-1.00E-5	-1.06E-5	-5.09E-6	-8.09E-6	-1.04E-5	-3.28E-6	-6.76E-6	-1.03E-5

Table 2.85: The coefficients c_j, $j = 0, \dots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{7}$, and $q = 200$

k=7 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.91E-3	4.98E-3	4.99E-3	4.98E-3	4.99E-3	4.99E-3	4.98E-3	4.99E-3	4.99E-3	4.98E-3	4.98E-3	4.99E-3
1	-4.90E-18	-6.75E-14	-3.05E-12	-2.54E-10	-9.22E-10	-1.85E-10	-4.69E-11	-8.86E-11	-2.58E-11	-7.94E-12	-9.82E-12	-7.10E-12
2	-4.78E-15	-1.76E-11	-5.94E-10	-1.17E-9	-1.10E-8	-8.82E-9	-2.70E-10	-2.06E-9	-2.21E-9	-5.85E-11	-4.24E-10	-9.73E-10
3	-7.69E-13	-1.08E-9	-3.72E-8	-5.65E-9	-6.40E-8	-2.53E-7	-1.59E-9	-2.31E-8	-9.36E-8	-4.23E-10	-8.48E-9	-5.24E-8
4	2.47E-24	-2.28E-8	-9.82E-7	-2.91E-8	-2.32E-7	-2.14E-6	-1.01E-8	-1.27E-7	-1.50E-6	-3.26E-9	-7.26E-8	-1.16E-6
5	-6.79E-11	-1.78E-7	-3.50E-6	-1.64E-7	-7.41E-7	-3.62E-6	-7.11E-8	-4.85E-7	-3.60E-6	-2.87E-8	-3.40E-7	-3.55E-6
6	-1.12E-8	-9.33E-7	-4.99E-6	-1.05E-6	-2.58E-6	-5.18E-6	-5.92E-7	-1.90E-6	-5.07E-6	-3.10E-7	-1.47E-6	-5.02E-6
7	3.70E-24	-6.62E-6	-1.06E-5	-9.97E-6	-1.23E-5	-1.11E-5	-7.18E-6	-1.03E-5	-1.09E-5	-5.04E-6	-8.80E-6	-1.07E-5

Table 2.86: The coefficients c_j, $j = 0, \dots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{8}$, and $q = 200$

k=8 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.49E-3	4.98E-3	4.99E-3	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	4.99E-3
1	3.41E-22	-9.66E-15	-4.29E-13	-2.03E-10	-6.10E-10	-8.50E-11	-3.71E-11	-5.25E-11	-9.06E-12	-6.18E-12	-5.03E-12	-1.74E-12
2	-3.82E-22	-2.54E-12	-8.17E-11	-8.08E-10	-6.52E-9	-2.97E-9	-1.81E-10	-1.00E-9	-5.67E-10	-3.81E-11	-1.67E-10	-1.89E-10
3	1.89E-26	-1.65E-10	-4.75E-9	-3.24E-9	-3.71E-8	-6.97E-8	-8.75E-10	-1.07E-8	-1.94E-8	-2.24E-10	-3.07E-9	-8.60E-9
4	-2.77E-27	-4.59E-9	-1.62E-7	-1.36E-8	-1.32E-7	-9.51E-7	-4.42E-9	-6.26E-8	-4.33E-7	-1.36E-9	-2.95E-8	-2.48E-7
5	4.60E-27	-5.17E-8	-1.85E-6	-6.02E-8	-3.77E-7	-2.85E-6	-2.38E-8	-2.29E-7	-2.47E-6	-8.83E-9	-1.46E-7	-2.13E-6
6	1.68E-26	-2.78E-7	-3.62E-6	-2.85E-7	-1.06E-6	-3.59E-6	-1.40E-7	-7.34E-7	-3.61E-6	-6.41E-8	-5.33E-7	-3.62E-6
7	6.86E-26	-1.27E-6	-5.13E-6	-1.51E-6	-3.32E-6	-5.37E-6	-9.41E-7	-2.56E-6	-5.25E-6	-5.51E-7	-2.04E-6	-5.17E-6
8	1.46E-23	-8.06E-6	-1.10E-5	-1.26E-5	-1.44E-5	-1.15E-5	-9.36E-6	-1.22E-5	-1.13E-5	-6.90E-6	-1.07E-5	-1.11E-5

Table 2.87: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 3}$, and $q = 400$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.40E-3	2.47E-3	2.49E-3	2.49E-3	2.49E-3	2.49E-3	2.49E-3	2.48E-3	2.49E-3	2.49E-3	2.47E-3	2.49E-3
1	-3.23E-17	-6.60E-11	-7.51E-9	-2.92E-10	-3.85E-9	-1.81E-8	-5.75E-11	-8.96E-10	-9.70E-9	-1.08E-11	-2.64E-10	-7.96E-9
2	-3.72E-14	-7.85E-9	-5.52E-7	-5.95E-9	-4.50E-8	-6.57E-7	-1.47E-9	-2.24E-8	-5.82E-7	-3.40E-10	-1.34E-8	-5.59E-7
3	-1.79E-11	-1.54E-7	-1.65E-6	-2.67E-7	-4.83E-7	-1.71E-6	-9.72E-8	-2.92E-7	-1.66E-6	-3.01E-8	-2.09E-7	-1.65E-6

Table 2.88: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 4}$, and $q = 400$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.44E-3	2.48E-3	2.49E-3	2.49E-3	2.49E-3	2.50E-3	2.49E-3	2.49E-3	2.49E-3	2.49E-3	2.49E-3	2.49E-3
1	-2.03E-18	-6.11E-12	-5.74E-10	-1.77E-10	-2.06E-9	-3.05E-9	-3.49E-11	-3.48E-10	-1.04E-9	-6.49E-12	-6.85E-11	-6.71E-10
2	-2.04E-15	-1.17E-9	-1.17E-7	-1.87E-9	-2.05E-8	-2.38E-7	-4.58E-10	-8.20E-9	-1.52E-7	-1.05E-10	-3.65E-9	-1.25E-7
3	-3.99E-13	-2.51E-8	-8.87E-7	-2.76E-8	-1.21E-7	-9.55E-7	-8.88E-9	-6.74E-8	-9.14E-7	-2.54E-9	-4.34E-8	-8.94E-7
4	-1.01E-10	-3.16E-7	-1.90E-6	-7.03E-7	-1.03E-6	-2.02E-6	-3.49E-7	-6.60E-7	-1.94E-6	-1.46E-7	-4.70E-7	-1.91E-6

Table 2.89: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 5}$, and $q = 400$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.46E-3	2.49E-3	2.50E-3	2.49E-3	2.50E-3	2.50E-3	2.49E-3	2.49E-3	2.50E-3	2.49E-3	2.49E-3	2.50E-3
1	-1.72E-19	-6.74E-13	-5.67E-11	-1.21E-10	-1.23E-9	-7.79E-10	-2.39E-11	-1.65E-10	-1.74E-10	-4.41E-12	-2.41E-11	-8.04E-11
2	-1.67E-16	-1.59E-10	-1.28E-8	-8.54E-10	-1.11E-8	-6.09E-8	-2.06E-10	-3.54E-9	-2.44E-8	-4.67E-11	-1.16E-9	-1.55E-8
3	-2.68E-14	-5.50E-9	-3.97E-7	-7.25E-9	-5.25E-8	-6.01E-7	-2.19E-9	-2.67E-8	-4.80E-7	-6.07E-10	-1.48E-8	-4.21E-7
4	-2.40E-12	-5.39E-8	-9.91E-7	-7.75E-8	-2.44E-7	-1.03E-6	-3.14E-8	-1.46E-7	-1.00E-6	-1.12E-8	-9.73E-8	-9.95E-7
5	-4.31E-10	-5.55E-7	-2.15E-6	-1.25E-6	-1.69E-6	-2.31E-6	-7.57E-7	-1.19E-6	-2.21E-6	-4.05E-7	-8.76E-7	-2.17E-6

Table 2.90: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 6}$, and $q = 400$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.47E-3	2.49E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.49E-3	2.50E-3	2.50E-3	2.49E-3	2.49E-3	2.50E-3
1	-1.78E-20	-8.52E-14	-6.69E-12	-9.01E-11	-7.92E-10	-2.69E-10	-1.76E-11	-8.90E-11	-4.17E-11	-3.23E-12	-1.05E-11	-1.32E-11
2	-1.72E-17	-2.17E-11	-1.39E-9	-4.79E-10	-6.66E-9	-1.55E-8	-1.14E-10	-1.69E-9	-4.27E-9	-2.54E-11	-4.16E-10	-2.07E-9
3	-2.66E-15	-1.08E-9	-8.50E-8	-2.87E-9	-2.83E-8	-2.97E-7	-8.29E-10	-1.27E-8	-1.60E-7	-2.23E-10	-5.83E-9	-1.07E-7
4	-1.79E-13	-1.43E-8	-6.42E-7	-1.95E-8	-1.01E-7	-7.50E-7	-7.11E-9	-5.74E-8	-7.09E-7	-2.35E-9	-3.54E-8	-6.68E-7
5	-1.05E-11	-9.51E-8	-1.04E-6	-1.57E-7	-4.07E-7	-1.10E-6	-7.64E-8	-2.62E-7	-1.06E-6	-3.32E-8	-1.80E-7	-1.04E-6
6	-1.51E-9	-8.60E-7	-2.35E-6	-1.85E-6	-2.34E-6	-2.53E-6	-1.25E-6	-1.77E-6	-2.43E-6	-7.81E-7	-1.38E-6	-2.38E-6

Table 2.91: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{7}$, and $q = 400$

k=7 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	1.39E-3	2.49E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.49E-3	2.50E-3	2.50E-3
1	3.10E-18	-1.20E-14	-8.99E-13	-7.03E-11	-5.39E-10	-1.15E-10	-1.37E-11	-5.30E-11	-1.33E-11	-2.49E-12	-5.32E-12	-2.90E-12
2	-4.31E-17	-3.12E-12	-1.77E-10	-3.06E-10	-4.25E-9	-4.77E-9	-7.12E-11	-8.75E-10	-9.75E-10	-1.57E-11	-1.69E-10	-3.56E-10
3	-1.24E-18	-1.87E-10	-1.12E-8	-1.43E-9	-1.71E-8	-1.11E-7	-3.99E-10	-6.63E-9	-3.74E-8	-1.04E-10	-2.43E-9	-1.83E-8
4	-3.53E-25	-3.71E-9	-2.57E-7	-7.27E-9	-5.33E-8	-5.46E-7	-2.47E-9	-2.82E-8	-4.11E-7	-7.74E-10	-1.56E-8	-3.17E-7
5	2.21E-27	-2.74E-8	-7.43E-7	-4.06E-8	-1.66E-7	-7.64E-7	-1.73E-8	-1.01E-7	-7.59E-7	-6.71E-9	-6.58E-8	-7.52E-7
6	2.50E-27	-1.49E-7	-1.09E-6	-2.60E-7	-5.92E-7	-1.18E-6	-1.45E-7	-4.07E-7	-1.13E-6	-7.33E-8	-2.92E-7	-1.11E-6
7	4.34E-25	-1.20E-6	-2.50E-6	-2.48E-6	-2.92E-6	-2.68E-6	-1.77E-6	-2.34E-6	-2.59E-6	-1.22E-6	-1.90E-6	-2.54E-6

Table 2.92: The coefficients c_j, $j = 0, \ldots, k$ of the partial fraction representation defined by (1.35) of (q, δ, α, k)-BURA approximation for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{8}$, and $q = 400$

k=8 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.49E-3	2.49E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3
1	1.1E-16	-1.8E-15	-1.33E-13	-5.69E-11	-3.83E-10	-5.74E-11	-1.10E-11	-3.40E-11	-5.27E-12	-1.99E-12	-3.02E-12	-8.25E-13
2	-1.8E-16	-4.8E-13	-2.55E-11	-2.13E-10	-2.84E-9	-1.77E-9	-4.87E-11	-4.88E-10	-2.82E-10	-1.05E-11	-7.69E-11	-7.81E-11
3	-7.78E-14	-3.10E-11	-1.51E-9	-8.27E-10	-1.11E-8	-3.73E-8	-2.24E-10	-3.65E-9	-8.91E-9	-5.70E-11	-1.06E-9	-3.38E-9
4	-2.42E-16	-8.24E-10	-5.08E-8	-3.42E-9	-3.20E-8	-3.26E-7	-1.10E-9	-1.55E-8	-1.61E-7	-3.30E-10	-7.43E-9	-8.82E-8
5	-1.48E-12	-8.53E-9	-4.51E-7	-1.50E-8	-8.61E-8	-6.20E-7	-5.84E-9	-4.97E-8	-5.83E-7	-2.10E-9	-3.03E-8	-5.20E-7
6	-4.00E-11	-4.44E-8	-7.61E-7	-7.09E-8	-2.44E-7	-7.83E-7	-3.43E-8	-1.57E-7	-7.68E-7	-1.52E-8	-1.07E-7	-7.63E-7
7	-1.57E-9	-2.15E-7	-1.15E-6	-3.77E-7	-7.79E-7	-1.26E-6	-2.31E-7	-5.69E-7	-1.21E-6	-1.32E-7	-4.24E-7	-1.17E-6
8	-1.30E-7	-1.55E-6	-2.62E-6	-3.13E-6	-3.45E-6	-2.80E-6	-2.31E-6	-2.85E-6	-2.72E-6	-1.68E-6	-2.39E-6	-2.66E-6

2.5 Tables type (e) for 0-URA-poles

Table 2.93: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{3}$, and $q = 1$

k=3 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.23E-5	-8.41E-4	-5.35E-3	-8.32E-5	-9.92E-4	-5.43E-3	-4.06E-5	-8.89E-4	-5.36E-3	-2.51E-5	-8.57E-4	-5.35E-3
2	-3.80E-3	-3.98E-2	-1.27E-1	-8.05E-3	-4.21E-2	-1.27E-1	-5.96E-3	-4.05E-2	-1.27E-1	-4.93E-3	-4.00E-2	-1.27E-1
3	-2.60E-1	-7.04E-1	-1.02E00	-3.48E-1	-7.21E-1	-1.03E00	-3.09E-1	-7.10E-1	-1.02E00	-2.87E-1	-7.06E-1	-1.02E00

Table 2.94: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the best uniform rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{4}$, and $q = 1$

k=4 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-5.37E-7	-9.04E-5	-8.33E-4	-1.83E-5	-1.47E-4	-8.87E-4	-6.00E-6	-1.08E-4	-8.44E-4	-2.53E-6	-9.60E-5	-8.35E-4
2	-1.75E-4	-4.64E-3	-2.04E-2	-9.63E-4	-5.70E-3	-2.10E-2	-5.31E-4	-4.99E-3	-2.05E-2	-3.47E-4	-4.75E-3	-2.04E-2
3	-1.12E-2	-7.92E-2	-2.15E-1	-2.61E-2	-8.72E-2	-2.17E-1	-1.92E-2	-8.20E-2	-2.15E-1	-1.55E-2	-8.01E-2	-2.15E-1
4	-4.00E-1	-9.55E-1	-1.25E00	-5.87E-1	-1.00E00	-1.25E00	-5.08E-1	-9.71E-1	-1.25E00	-4.61E-1	-9.61E-1	-1.25E00

Table 2.95: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{5}$, and $q = 1$

k=5 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-3.20E-8	-1.22E-5	-1.59E-4	-6.90E-6	-3.53E-5	-1.92E-4	-1.71E-6	-1.93E-5	-1.66E-4	-5.12E-7	-1.44E-5	-1.60E-4
2	-1.08E-5	-6.45E-4	-3.93E-3	-2.12E-4	-1.08E-3	-4.29E-3	-8.75E-5	-7.98E-4	-4.01E-3	-4.34E-5	-6.96E-4	-3.95E-3
3	-7.18E-4	-1.17E-2	-4.25E-2	-4.02E-3	-1.54E-2	-4.46E-2	-2.34E-3	-1.31E-2	-4.30E-2	-1.55E-3	-1.22E-2	-4.26E-2
4	-2.24E-2	-1.24E-1	-3.02E-1	-5.75E-2	-1.44E-1	-3.09E-1	-4.23E-2	-1.32E-1	-3.04E-1	-3.37E-2	-1.27E-1	-3.02E-1
5	-5.46E-1	-1.20E00	-1.47E00	-8.77E-1	-1.30E00	-1.49E00	-7.46E-1	-1.24E00	-1.47E00	-6.65E-1	-1.22E00	-1.47E00

Table 2.96: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{6}$, and $q = 1$

k=6 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-2.42E-9	-1.95E-6	-3.52E-5	-3.50E-6	-1.27E-5	-5.48E-5	-7.25E-7	-5.03E-6	-3.98E-5	-1.71E-7	-2.90E-6	-3.61E-5
2	-8.33E-7	-1.05E-4	-8.73E-4	-7.03E-5	-2.88E-4	-1.09E-3	-2.30E-5	-1.70E-4	-9.27E-4	-8.84E-6	-1.27E-4	-8.84E-4
3	-5.70E-5	-1.95E-3	-9.51E-3	-9.66E-4	-3.58E-3	-1.08E-2	-4.50E-4	-2.59E-3	-9.84E-3	-2.40E-4	-2.19E-3	-9.58E-3
4	-1.82E-3	-2.15E-2	-6.94E-2	-1.06E-2	-3.08E-2	-7.48E-2	-6.47E-3	-2.54E-2	-7.08E-2	-4.35E-3	-2.30E-2	-6.97E-2
5	-3.68E-2	-1.73E-1	-3.86E-1	-1.03E-1	-2.12E-1	-4.01E-1	-7.61E-2	-1.90E-1	-3.90E-1	-6.02E-2	-1.79E-1	-3.87E-1
6	-6.94E-1	-1.45E00	-1.68E00	-1.22E00	-1.63E00	-1.73E00	-1.02E00	-1.53E00	-1.70E00	-8.99E-1	-1.48E00	-1.69E00

Table 2.97: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 7}$, and $q = 1$

k=7 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-2.22E-10	-3.58E-7	-8.74E-6	-2.11E-6	-6.03E-6	-2.05E-5	-3.90E-7	-1.84E-6	-1.16E-5	-7.85E-8	-7.89E-7	-9.40E-6
2	-7.71E-8	-1.93E-5	-2.17E-4	-3.09E-5	-1.02E-4	-3.44E-4	-8.41E-6	-4.76E-5	-2.52E-4	-2.62E-6	-2.92E-5	-2.25E-4
3	-5.35E-6	-3.66E-4	-2.37E-3	-3.21E-4	-1.06E-3	-3.16E-3	-1.23E-4	-6.43E-4	-2.60E-3	-5.36E-5	-4.72E-4	-2.43E-3
4	-1.75E-4	-4.14E-3	-1.74E-2	-2.82E-3	-8.27E-3	-2.08E-2	-1.42E-3	-5.95E-3	-1.84E-2	-7.99E-4	-4.87E-3	-1.76E-2
5	-3.56E-3	-3.37E-2	-9.93E-2	-2.19E-2	-5.20E-2	-1.11E-1	-1.38E-2	-4.22E-2	-1.03E-1	-9.39E-3	-3.72E-2	-1.00E-1
6	-5.37E-2	-2.23E-1	-4.66E-1	-1.61E-1	-2.90E-1	-4.95E-1	-1.20E-1	-2.55E-1	-4.75E-1	-9.51E-2	-2.37E-1	-4.68E-1
7	-8.45E-1	-1.69E00	-1.90E00	-1.60E00	-1.99E00	-1.98E00	-1.34E00	-1.83E00	-1.93E00	-1.16E00	-1.75E00	-1.91E00

Table 2.98: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 8}$, and $q = 1$

k=8 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-2.38E-11	-7.35E-8	-2.38E-6	-1.41E-6	-3.44E-6	-9.54E-6	-2.43E-7	-8.66E-7	-4.20E-6	-4.40E-8	-2.86E-7	-2.82E-6
2	-8.29E-9	-3.97E-6	-5.92E-5	-1.64E-5	-4.50E-5	-1.32E-4	-3.88E-6	-1.71E-5	-8.06E-5	-1.02E-6	-8.47E-6	-6.47E-5
3	-5.81E-7	-7.56E-5	-6.48E-4	-1.35E-4	-3.92E-4	-1.10E-3	-4.38E-5	-1.98E-4	-7.89E-4	-1.60E-5	-1.23E-4	-6.85E-4
4	-1.92E-5	-8.70E-4	-4.76E-3	-9.72E-4	-2.70E-3	-6.77E-3	-4.14E-4	-1.67E-3	-5.42E-3	-1.96E-4	-1.21E-3	-4.93E-3
5	-3.99E-4	-7.24E-3	-2.72E-2	-6.34E-3	-1.56E-2	-3.45E-2	-3.39E-3	-1.12E-2	-2.97E-2	-1.98E-3	-9.00E-3	-2.79E-2
6	-6.00E-3	-4.78E-2	-1.31E-1	-3.84E-2	-7.88E-2	-1.53E-1	-2.48E-2	-6.33E-2	-1.39E-1	-1.72E-2	-5.49E-2	-1.33E-1
7	-7.27E-2	-2.75E-1	-5.43E-1	-2.32E-1	-3.78E-1	-5.90E-1	-1.75E-1	-3.28E-1	-5.60E-1	-1.38E-1	-3.00E-1	-5.47E-1
8	-9.97E-1	-1.92E00	-2.12E00	-2.03E00	-2.38E00	-2.27E00	-1.68E00	-2.16E00	-2.17E00	-1.45E00	-2.04E00	-2.14E00

Table 2.99: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 3}$, and $q = 100$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-2.41E-6	-1.69E-4	-1.55E-3	-2.60E-5	-2.02E-4	-1.57E-3	-1.09E-5	-1.79E-4	-1.55E-3	-6.03E-6	-1.72E-4	-1.55E-3
2	-1.44E-3	-9.07E-3	-1.70E-2	-3.64E-3	-9.91E-3	-1.71E-2	-2.52E-3	-9.34E-3	-1.70E-2	-2.00E-3	-9.16E-3	-1.70E-2
3	-1.39E-1	-2.70E-1	-3.19E-1	-1.99E-1	-2.80E-1	-3.20E-1	-1.72E-1	-2.73E-1	-3.19E-1	-1.57E-1	-2.71E-1	-3.19E-1

Table 2.100: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 4}$, and $q = 100$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.40E-7	-3.64E-5	-5.45E-4	-7.16E-6	-5.61E-5	-5.73E-4	-2.11E-6	-4.27E-5	-5.51E-4	-8.11E-7	-3.84E-5	-5.46E-4
2	-6.34E-5	-1.17E-3	-4.22E-3	-4.56E-4	-1.50E-3	-4.33E-3	-2.30E-4	-1.28E-3	-4.24E-3	-1.41E-4	-1.20E-3	-4.23E-3
3	-5.44E-3	-2.54E-2	-4.39E-2	-1.48E-2	-2.92E-2	-4.48E-2	-1.03E-2	-2.67E-2	-4.40E-2	-8.03E-3	-2.59E-2	-4.39E-2
4	-2.36E-1	-4.32E-1	-5.15E-1	-3.71E-1	-4.63E-1	-5.21E-1	-3.13E-1	-4.43E-1	-5.16E-1	-2.79E-1	-4.36E-1	-5.15E-1

Table 2.101: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ of the type (1.46) on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 5}$, and $q = 100$

k=5 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.17E-8	-7.81E-6	-1.40E-4	-3.15E-6	-2.02E-5	-1.67E-4	-7.36E-7	-1.18E-5	-1.46E-4	-2.10E-7	-9.09E-6	-1.41E-4
2	-4.09E-6	-2.23E-4	-1.68E-3	-1.09E-4	-3.77E-4	-1.79E-3	-4.13E-5	-2.76E-4	-1.70E-3	-1.91E-5	-2.41E-4	-1.69E-3
3	-3.26E-4	-3.67E-3	-9.61E-3	-2.31E-3	-5.22E-3	-1.02E-2	-1.26E-3	-4.23E-3	-9.74E-3	-7.86E-4	-3.86E-3	-9.64E-3
4	-1.24E-2	-4.85E-2	-8.01E-2	-3.69E-2	-5.96E-2	-8.34E-2	-2.60E-2	-5.27E-2	-8.09E-2	-2.00E-2	-5.00E-2	-8.03E-2
5	-3.40E-1	-6.03E-1	-7.21E-1	-5.88E-1	-6.76E-1	-7.38E-1	-4.89E-1	-6.31E-1	-7.25E-1	-4.28E-1	-6.13E-1	-7.22E-1

Table 2.102: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ of the type (1.46) on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 6}$, and $q = 100$

k=6 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.20E-9	-1.59E-6	-3.38E-5	-1.77E-6	-9.04E-6	-5.21E-5	-3.59E-7	-3.88E-6	-3.81E-5	-8.39E-8	-2.32E-6	-3.47E-5
2	-3.53E-7	-5.32E-5	-6.32E-4	-3.94E-5	-1.34E-4	-7.57E-4	-1.20E-5	-8.24E-5	-6.64E-4	-4.31E-6	-6.34E-5	-6.39E-4
3	-2.57E-5	-7.00E-4	-3.21E-3	-5.77E-4	-1.36E-3	-3.58E-3	-2.50E-4	-9.53E-4	-3.31E-3	-1.26E-4	-7.91E-4	-3.23E-3
4	-9.41E-4	-7.95E-3	-1.83E-2	-6.86E-3	-1.25E-2	-2.03E-2	-3.94E-3	-9.79E-3	-1.89E-2	-2.52E-3	-8.63E-3	-1.85E-2
5	-2.21E-2	-7.66E-2	-1.23E-1	-7.12E-2	-1.01E-1	-1.31E-1	-5.08E-2	-8.70E-2	-1.25E-1	-3.89E-2	-8.05E-2	-1.23E-1
6	-4.50E-1	-7.80E-1	-9.33E-1	-8.48E-1	-9.19E-1	-9.73E-1	-6.99E-1	-8.40E-1	-9.43E-1	-6.04E-1	-8.03E-1	-9.35E-1

Table 2.103: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 7}$, and $q = 100$

k=7 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.39E-10	-3.27E-7	-8.62E-6	-1.14E-6	-4.82E-6	-2.00E-5	-2.12E-7	-1.58E-6	-1.15E-5	-4.35E-8	-7.04E-7	-9.26E-6
2	-3.80E-8	-1.33E-5	-1.94E-4	-1.86E-5	-5.98E-5	-2.97E-4	-4.77E-6	-3.01E-5	-2.23E-4	-1.42E-6	-1.94E-5	-2.01E-4
3	-2.53E-6	-1.68E-4	-1.37E-3	-2.01E-4	-4.79E-4	-1.68E-3	-7.19E-5	-2.90E-4	-1.46E-3	-2.96E-5	-2.14E-4	-1.39E-3
4	-8.85E-5	-1.61E-3	-5.57E-3	-1.85E-3	-3.54E-3	-6.73E-3	-8.80E-4	-2.42E-3	-5.91E-3	-4.69E-4	-1.93E-3	-5.65E-3
5	-2.02E-3	-1.40E-2	-3.00E-2	-1.52E-2	-2.41E-2	-3.51E-2	-9.12E-3	-1.85E-2	-3.15E-2	-5.96E-3	-1.58E-2	-3.04E-2
6	-3.41E-2	-1.08E-1	-1.70E-1	-1.18E-1	-1.53E-1	-1.88E-1	-8.51E-2	-1.29E-1	-1.75E-1	-6.53E-2	-1.17E-1	-1.71E-1
7	-5.64E-1	-9.60E-1	-1.15E00	-1.15E00	-1.19E00	-1.23E00	-9.41E-1	-1.07E00	-1.17E00	-8.06E-1	-1.01E00	-1.15E00

Table 2.104: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 8}$, and $q = 100$

k=8 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.76E-11	-7.05E-8	-2.37E-6	-7.99E-7	-2.93E-6	-9.43E-6	-1.41E-7	-7.86E-7	-4.17E-6	-2.65E-8	-2.68E-7	-2.80E-6
2	-4.81E-9	-3.29E-6	-5.69E-5	-1.04E-5	-3.09E-5	-1.24E-4	-2.36E-6	-1.28E-5	-7.67E-5	-6.03E-7	-6.68E-6	-6.19E-5
3	-2.98E-7	-4.57E-5	-5.23E-4	-8.81E-5	-2.09E-4	-8.14E-4	-2.70E-5	-1.10E-4	-6.19E-4	-9.39E-6	-7.12E-5	-5.48E-4
4	-9.89E-6	-3.91E-4	-2.27E-3	-6.55E-4	-1.27E-3	-2.99E-3	-2.63E-4	-7.65E-4	-2.51E-3	-1.18E-4	-5.46E-4	-2.34E-3
5	-2.20E-4	-3.04E-3	-9.04E-3	-4.45E-3	-7.41E-3	-1.19E-2	-2.26E-3	-5.04E-3	-9.99E-3	-1.26E-3	-3.91E-3	-9.29E-3
6	-3.62E-3	-2.17E-2	-4.42E-2	-2.81E-2	-4.04E-2	-5.44E-2	-1.75E-2	-3.08E-2	-4.77E-2	-1.17E-2	-2.58E-2	-4.51E-2
7	-4.82E-2	-1.43E-1	-2.20E-1	-1.76E-1	-2.16E-1	-2.54E-1	-1.29E-1	-1.80E-1	-2.32E-1	-9.93E-2	-1.60E-1	-2.23E-1
8	-6.79E-1	-1.14E00	-1.37E00	-1.49E00	-1.50E00	-1.51E00	-1.21E00	-1.33E00	-1.42E00	-1.03E00	-1.23E00	-1.38E00

Table 2.105: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ of the type (1.46) on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k}=\mathbf{3}$, and $q = 200$

k=3 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.95E-6	-1.08E-4	-8.95E-4	-2.40E-5	-1.34E-4	-9.05E-4	-9.77E-6	-1.16E-4	-8.97E-4	-5.23E-6	-1.11E-4	-8.95E-4
2	-1.40E-3	-8.52E-3	-1.54E-2	-3.57E-3	-9.35E-3	-1.56E-2	-2.46E-3	-8.79E-3	-1.54E-2	-1.95E-3	-8.61E-3	-1.54E-2
3	-1.38E-1	-2.67E-1	-3.16E-1	-1.98E-1	-2.77E-1	-3.18E-1	-1.71E-1	-2.70E-1	-3.16E-1	-1.56E-1	-2.68E-1	-3.16E-1

Table 2.106: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k}=\mathbf{4}$, and $q = 200$

k=4 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.03E-7	-2.36E-5	-3.87E-4	-6.57E-6	-3.74E-5	-4.04E-4	-1.86E-6	-2.80E-5	-3.91E-4	-6.84E-7	-2.50E-5	-3.88E-4
2	-5.98E-5	-1.00E-3	-3.12E-3	-4.44E-4	-1.32E-3	-3.23E-3	-2.22E-4	-1.11E-3	-3.14E-3	-1.35E-4	-1.03E-3	-3.13E-3
3	-5.36E-3	-2.47E-2	-4.24E-2	-1.47E-2	-2.85E-2	-4.33E-2	-1.02E-2	-2.60E-2	-4.26E-2	-7.93E-3	-2.51E-2	-4.25E-2
4	-2.34E-1	-4.29E-1	-5.12E-1	-3.69E-1	-4.60E-1	-5.18E-1	-3.11E-1	-4.40E-1	-5.13E-1	-2.78E-1	-4.32E-1	-5.13E-1

Table 2.107: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k}=\mathbf{5}$, and $q = 200$

k=5 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-8.24E-9	-5.76E-6	-1.23E-4	-2.88E-6	-1.46E-5	-1.45E-4	-6.49E-7	-8.62E-6	-1.28E-4	-1.77E-7	-6.68E-6	-1.24E-4
2	-3.70E-6	-1.71E-4	-1.10E-3	-1.06E-4	-3.06E-4	-1.17E-3	-3.95E-5	-2.17E-4	-1.12E-3	-1.80E-5	-1.86E-4	-1.10E-3
3	-3.16E-4	-3.41E-3	-8.55E-3	-2.27E-3	-4.93E-3	-9.14E-3	-1.23E-3	-3.96E-3	-8.68E-3	-7.69E-4	-3.60E-3	-8.57E-3
4	-1.23E-2	-4.76E-2	-7.88E-2	-3.67E-2	-5.87E-2	-8.20E-2	-2.58E-2	-5.18E-2	-7.95E-2	-1.98E-2	-4.91E-2	-7.89E-2
5	-3.39E-1	-6.00E-1	-7.18E-1	-5.85E-1	-6.72E-1	-7.35E-1	-4.87E-1	-6.28E-1	-7.22E-1	-4.26E-1	-6.09E-1	-7.19E-1

Table 2.108: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k}=\mathbf{6}$, and $q = 200$

k=6 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-8.56E-10	-1.34E-6	-3.24E-5	-1.62E-6	-7.10E-6	-4.94E-5	-3.17E-7	-3.16E-6	-3.64E-5	-7.10E-8	-1.93E-6	-3.33E-5
2	-3.04E-7	-3.93E-5	-4.74E-4	-3.81E-5	-1.05E-4	-5.55E-4	-1.13E-5	-6.23E-5	-4.95E-4	-4.01E-6	-4.72E-5	-4.79E-4
3	-2.44E-5	-6.05E-4	-2.40E-3	-5.67E-4	-1.23E-3	-2.75E-3	-2.44E-4	-8.43E-4	-2.49E-3	-1.22E-4	-6.90E-4	-2.42E-3
4	-9.22E-4	-7.61E-3	-1.73E-2	-6.80E-3	-1.21E-2	-1.94E-2	-3.89E-3	-9.44E-3	-1.79E-2	-2.49E-3	-8.29E-3	-1.75E-2
5	-2.19E-2	-7.56E-2	-1.21E-1	-7.09E-2	-1.00E-1	-1.30E-1	-5.05E-2	-8.60E-2	-1.24E-1	-3.87E-2	-7.95E-2	-1.22E-1
6	-4.48E-1	-7.76E-1	-9.30E-1	-8.45E-1	-9.15E-1	-9.71E-1	-6.96E-1	-8.36E-1	-9.41E-1	-6.02E-1	-7.99E-1	-9.32E-1

Table 2.109: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{7}$, and $q = 200$

k=7 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.05E-10	-3.00E-7	-8.50E-6	-1.04E-6	-4.03E-6	-1.95E-5	-1.88E-7	-1.39E-6	-1.13E-5	-3.72E-8	-6.34E-7	-9.12E-6
2	-3.15E-8	-1.04E-5	-1.73E-4	-1.79E-5	-4.70E-5	-2.55E-4	-4.51E-6	-2.33E-5	-1.96E-4	-1.31E-6	-1.50E-5	-1.78E-4
3	-2.33E-6	-1.34E-4	-9.56E-4	-1.97E-4	-4.17E-4	-1.19E-3	-6.99E-5	-2.42E-4	-1.02E-3	-2.85E-5	-1.75E-4	-9.72E-4
4	-8.55E-5	-1.47E-3	-4.74E-3	-1.83E-3	-3.36E-3	-5.91E-3	-8.67E-4	-2.26E-3	-5.08E-3	-4.60E-4	-1.78E-3	-4.82E-3
5	-1.99E-3	-1.36E-2	-2.91E-2	-1.51E-2	-2.36E-2	-3.41E-2	-9.05E-3	-1.81E-2	-3.06E-2	-5.91E-3	-1.54E-2	-2.94E-2
6	-3.39E-2	-1.07E-1	-1.68E-1	-1.17E-1	-1.52E-1	-1.87E-1	-8.47E-2	-1.28E-1	-1.74E-1	-6.50E-2	-1.16E-1	-1.70E-1
7	-5.61E-1	-9.56E-1	-1.15E00	-1.14E00	-1.19E00	-1.23E00	-9.38E-1	-1.07E00	-1.17E00	-8.04E-1	-1.00E00	-1.15E00

Table 2.110: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ of the type (1.46) on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{8}$, and $q = 200$

k=8 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.41E-11	-6.76E-8	-2.36E-6	-7.31E-7	-2.55E-6	-9.32E-6	-1.25E-7	-7.20E-7	-4.14E-6	-2.28E-8	-2.52E-7	-2.78E-6
2	-3.90E-9	-2.81E-6	-5.45E-5	-1.00E-5	-2.50E-5	-1.16E-4	-2.23E-6	-1.04E-5	-7.29E-5	-5.56E-7	-5.57E-6	-5.92E-5
3	-2.65E-7	-3.57E-5	-4.21E-4	-8.62E-5	-1.77E-4	-6.22E-4	-2.62E-5	-8.93E-5	-4.88E-4	-8.98E-6	-5.65E-5	-4.39E-4
4	-9.36E-6	-3.35E-4	-1.67E-3	-6.47E-4	-1.18E-3	-2.33E-3	-2.58E-4	-6.87E-4	-1.88E-3	-1.16E-4	-4.80E-4	-1.73E-3
5	-2.14E-4	-2.87E-3	-8.25E-3	-4.41E-3	-7.18E-3	-1.11E-2	-2.23E-3	-4.83E-3	-9.20E-3	-1.24E-3	-3.72E-3	-8.50E-3
6	-3.57E-3	-2.12E-2	-4.33E-2	-2.80E-2	-3.99E-2	-5.35E-2	-1.74E-2	-3.03E-2	-4.68E-2	-1.16E-2	-2.53E-2	-4.42E-2
7	-4.79E-2	-1.42E-1	-2.19E-1	-1.76E-1	-2.15E-1	-2.52E-1	-1.28E-1	-1.79E-1	-2.30E-1	-9.89E-2	-1.59E-1	-2.22E-1
8	-6.77E-1	-1.14E00	-1.36E00	-1.48E00	-1.49E00	-1.51E00	-1.21E00	-1.32E00	-1.41E00	-1.03E00	-1.22E00	-1.38E00

Table 2.111: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ of the type (1.46) on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{3}$, and $q = 400$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.70E-6	-7.38E-5	-5.07E-4	-2.29E-5	-9.56E-5	-5.14E-4	-9.16E-6	-8.06E-5	-5.09E-4	-4.80E-6	-7.59E-5	-5.07E-4
2	-1.38E-3	-8.24E-3	-1.47E-2	-3.53E-3	-9.06E-3	-1.48E-2	-2.43E-3	-8.51E-3	-1.47E-2	-1.92E-3	-8.33E-3	-1.47E-2
3	-1.38E-1	-2.65E-1	-3.15E-1	-1.97E-1	-2.76E-1	-3.16E-1	-1.70E-1	-2.69E-1	-3.15E-1	-1.56E-1	-2.67E-1	-3.15E-1

Table 2.112: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^{\alpha}/(1+q\,t^{\alpha})$ with functions from $\mathcal{R}_k$ of the type (1.46) on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{4}$, and $q = 400$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-8.26E-8	-1.49E-5	-2.42E-4	-6.25E-6	-2.52E-5	-2.51E-4	-1.73E-6	-1.81E-5	-2.44E-4	-6.14E-7	-1.59E-5	-2.42E-4
2	-5.80E-5	-9.14E-4	-2.60E-3	-4.38E-4	-1.22E-3	-2.70E-3	-2.18E-4	-1.01E-3	-2.62E-3	-1.32E-4	-9.46E-4	-2.60E-3
3	-5.31E-3	-2.43E-2	-4.17E-2	-1.46E-2	-2.81E-2	-4.27E-2	-1.01E-2	-2.56E-2	-4.19E-2	-7.87E-3	-2.48E-2	-4.18E-2
4	-2.34E-1	-4.27E-1	-5.11E-1	-3.68E-1	-4.58E-1	-5.17E-1	-3.10E-1	-4.38E-1	-5.12E-1	-2.77E-1	-4.31E-1	-5.11E-1

Table 2.113: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ of the type (1.46) on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=5}$, and $q = 400$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-6.07E-9	-3.85E-6	-9.78E-5	-2.74E-6	-1.01E-5	-1.13E-4	-5.99E-7	-5.82E-6	-1.01E-4	-1.57E-7	-4.48E-6	-9.85E-5
2	-3.49E-6	-1.42E-4	-7.46E-4	-1.04E-4	-2.68E-4	-8.13E-4	-3.85E-5	-1.84E-4	-7.61E-4	-1.74E-5	-1.56E-4	-7.49E-4
3	-3.11E-4	-3.28E-3	-8.06E-3	-2.26E-3	-4.78E-3	-8.66E-3	-1.22E-3	-3.83E-3	-8.20E-3	-7.61E-4	-3.47E-3	-8.09E-3
4	-1.22E-2	-4.72E-2	-7.81E-2	-3.66E-2	-5.82E-2	-8.13E-2	-2.57E-2	-5.14E-2	-7.88E-2	-1.97E-2	-4.87E-2	-7.82E-2
5	-3.38E-1	-5.98E-1	-7.17E-1	-5.84E-1	-6.70E-1	-7.34E-1	-4.86E-1	-6.26E-1	-7.21E-1	-4.25E-1	-6.08E-1	-7.18E-1

Table 2.114: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ of the type (1.46) on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=6}$, and $q = 400$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-6.05E-10	-1.03E-6	-2.98E-5	-1.53E-6	-5.18E-6	-4.45E-5	-2.93E-7	-2.34E-6	-3.34E-5	-6.31E-8	-1.45E-6	-3.06E-5
2	-2.76E-7	-2.96E-5	-3.19E-4	-3.73E-5	-8.74E-5	-3.71E-4	-1.10E-5	-4.92E-5	-3.32E-4	-3.85E-6	-3.62E-5	-3.21E-4
3	-2.37E-5	-5.54E-4	-2.01E-3	-5.62E-4	-1.17E-3	-2.37E-3	-2.41E-4	-7.86E-4	-2.10E-3	-1.20E-4	-6.36E-4	-2.03E-3
4	-9.12E-4	-7.45E-3	-1.69E-2	-6.77E-3	-1.19E-2	-1.89E-2	-3.87E-3	-9.26E-3	-1.74E-2	-2.47E-3	-8.12E-3	-1.70E-2
5	-2.18E-2	-7.51E-2	-1.21E-1	-7.07E-2	-9.95E-2	-1.29E-1	-5.03E-2	-8.55E-2	-1.23E-1	-3.85E-2	-7.90E-2	-1.21E-1
6	-4.47E-1	-7.74E-1	-9.29E-1	-8.43E-1	-9.13E-1	-9.69E-1	-6.95E-1	-8.34E-1	-9.39E-1	-6.01E-1	-7.97E-1	-9.31E-1

Table 2.115: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ of the type (1.46) on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=7}$, and $q = 400$

k=7 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-7.44E-11	-2.58E-7	-8.25E-6	-9.87E-7	-3.10E-6	-1.86E-5	-1.73E-7	-1.12E-6	-1.09E-5	-3.31E-8	-5.30E-7	-8.85E-6
2	-2.73E-8	-7.72E-6	-1.39E-4	-1.75E-5	-3.81E-5	-1.94E-4	-4.37E-6	-1.80E-5	-1.55E-4	-1.25E-6	-1.13E-5	-1.42E-4
3	-2.21E-6	-1.14E-4	-6.80E-4	-1.95E-4	-3.83E-4	-8.89E-4	-6.88E-5	-2.16E-4	-7.38E-4	-2.79E-5	-1.52E-4	-6.94E-4
4	-8.39E-5	-1.40E-3	-4.37E-3	-1.82E-3	-3.27E-3	-5.55E-3	-8.60E-4	-2.18E-3	-4.71E-3	-4.55E-4	-1.71E-3	-4.45E-3
5	-1.97E-3	-1.34E-2	-2.86E-2	-1.51E-2	-2.34E-2	-3.37E-2	-9.01E-3	-1.79E-2	-3.01E-2	-5.88E-3	-1.52E-2	-2.90E-2
6	-3.38E-2	-1.07E-1	-1.68E-1	-1.17E-1	-1.52E-1	-1.86E-1	-8.45E-2	-1.28E-1	-1.73E-1	-6.48E-2	-1.16E-1	-1.69E-1
7	-5.60E-1	-9.54E-1	-1.14E00	-1.14E00	-1.19E00	-1.23E00	-9.37E-1	-1.07E00	-1.17E00	-8.02E-1	-1.00E00	-1.15E00

Table 2.116: The poles of $\bar{r}_{q,\delta,\alpha,k}(t)$, (1.46), of (q,δ,α,k)-0-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ of the type (1.46) on $[\delta,1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k=8}$, and $q = 400$

k=8 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.05E-11	-6.26E-8	-2.33E-6	-6.92E-7	-2.06E-6	-9.09E-6	-1.15E-7	-6.17E-7	-4.08E-6	-2.03E-8	-2.25E-7	-2.75E-6
2	-3.25E-9	-2.22E-6	-4.99E-5	-9.83E-6	-2.02E-5	-1.00E-4	-2.16E-6	-8.17E-6	-6.56E-5	-5.29E-7	-4.34E-6	-5.40E-5
3	-2.45E-7	-2.83E-5	-3.03E-4	-8.53E-5	-1.59E-4	-4.43E-4	-2.57E-5	-7.62E-5	-3.48E-4	-8.76E-6	-4.65E-5	-3.15E-4
4	-9.07E-6	-3.05E-4	-1.36E-3	-6.42E-4	-1.13E-3	-2.03E-3	-2.56E-4	-6.46E-4	-1.57E-3	-1.14E-4	-4.44E-4	-1.42E-3
5	-2.11E-4	-2.78E-3	-7.90E-3	-4.39E-3	-7.06E-3	-1.08E-2	-2.22E-3	-4.73E-3	-8.86E-3	-1.24E-3	-3.62E-3	-8.15E-3
6	-3.55E-3	-2.10E-2	-4.28E-2	-2.79E-2	-3.96E-2	-5.31E-2	-1.73E-2	-3.00E-2	-4.63E-2	-1.15E-2	-2.51E-2	-4.38E-2
7	-4.78E-2	-1.41E-1	-2.18E-1	-1.75E-1	-2.14E-1	-2.52E-1	-1.28E-1	-1.78E-1	-2.30E-1	-9.87E-2	-1.58E-1	-2.21E-1
8	-6.76E-1	-1.14E00	-1.36E00	-1.48E00	-1.49E00	-1.51E00	-1.21E00	-1.32E00	-1.41E00	-1.03E00	-1.22E00	-1.38E00

2.6 Tables type (f) for 0-URA-decomposition coefficients

Table 2.117: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{3}$, and $q = 1$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	5.52E-1	6.95E-1	8.60E-1	5.64E-1	6.97E-1	8.60E-1	5.59E-1	6.96E-1	8.60E-1	5.56E-1	6.95E-1	8.60E-1
1	-1.13E-6	-3.83E-5	-9.73E-5	-8.38E-6	-4.70E-5	-9.96E-5	-3.94E-6	-4.10E-5	-9.77E-5	-2.38E-6	-3.92E-5	-9.74E-5
2	-7.03E-4	-7.62E-3	-2.11E-2	-1.43E-3	-8.14E-3	-2.13E-2	-1.08E-3	-7.79E-3	-2.11E-2	-9.02E-4	-7.67E-3	-2.11E-2
3	-6.86E-2	-3.21E-1	-6.91E-1	-8.66E-2	-3.27E-1	-6.92E-1	-7.86E-2	-3.23E-1	-6.91E-1	-7.42E-2	-3.22E-1	-6.91E-1

Table 2.118: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{4}$, and $q = 1$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	5.70E-1	7.25E-1	8.86E-1	5.87E-1	7.30E-1	8.86E-1	5.80E-1	7.27E-1	8.86E-1	5.76E-1	7.26E-1	8.86E-1
1	-2.36E-8	-1.36E-6	-3.70E-6	-1.10E-6	-2.53E-6	-4.06E-6	-3.25E-7	-1.71E-6	-3.77E-6	-1.27E-7	-1.47E-6	-3.71E-6
2	-1.83E-5	-3.23E-4	-7.14E-4	-1.06E-4	-4.19E-4	-7.43E-4	-5.77E-5	-3.55E-4	-7.20E-4	-3.72E-5	-3.34E-4	-7.16E-4
3	-1.96E-3	-1.67E-2	-4.82E-2	-4.23E-3	-1.86E-2	-4.91E-2	-3.20E-3	-1.73E-2	-4.84E-2	-2.64E-3	-1.69E-2	-4.82E-2
4	-9.68E-2	-4.10E-1	-7.76E-1	-1.32E-1	-4.25E-1	-7.78E-1	-1.17E-1	-4.15E-1	-7.77E-1	-1.09E-1	-4.12E-1	-7.76E-1

Table 2.119: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{5}$, and $q = 1$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	5.83E-1	7.48E-1	9.02E-1	6.06E-1	7.55E-1	9.03E-1	5.98E-1	7.51E-1	9.03E-1	5.93E-1	7.49E-1	9.02E-1
1	-7.10E-10	-6.76E-8	-2.03E-7	-2.85E-7	-2.72E-7	-2.69E-7	-5.87E-8	-1.22E-7	-2.16E-7	-1.52E-8	-8.40E-8	-2.05E-7
2	-6.13E-7	-1.71E-5	-3.80E-5	-1.52E-5	-3.35E-5	-4.34E-5	-5.91E-6	-2.26E-5	-3.92E-5	-2.77E-6	-1.89E-5	-3.82E-5
3	-7.88E-5	-1.02E-3	-2.19E-3	-4.45E-4	-1.45E-3	-2.36E-3	-2.61E-4	-1.18E-3	-2.23E-3	-1.72E-4	-1.08E-3	-2.20E-3
4	-3.69E-3	-2.74E-2	-8.26E-2	-8.48E-3	-3.22E-2	-8.56E-2	-6.49E-3	-2.92E-2	-8.33E-2	-5.31E-3	-2.81E-2	-8.27E-2
5	-1.24E-1	-4.89E-1	-8.31E-1	-1.83E-1	-5.20E-1	-8.34E-1	-1.60E-1	-5.01E-1	-8.32E-1	-1.46E-1	-4.93E-1	-8.31E-1

Table 2.120: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{6}$, and $q = 1$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	5.95E-1	7.65E-1	9.14E-1	6.22E-1	7.76E-1	9.16E-1	6.13E-1	7.70E-1	9.15E-1	6.07E-1	7.67E-1	9.15E-1
1	-2.85E-11	-4.34E-9	-1.45E-8	-1.09E-7	-5.22E-8	-2.84E-8	-1.77E-8	-1.51E-8	-1.74E-8	-3.38E-9	-7.27E-9	-1.51E-8
2	-2.60E-8	-1.12E-6	-2.70E-6	-3.52E-6	-4.29E-6	-3.80E-6	-1.03E-6	-2.13E-6	-2.96E-6	-3.58E-7	-1.45E-6	-2.75E-6
3	-3.72E-6	-7.18E-5	-1.46E-4	-7.48E-5	-1.57E-4	-1.79E-4	-3.37E-5	-1.04E-4	-1.54E-4	-1.74E-5	-8.31E-5	-1.48E-4
4	-2.02E-4	-2.18E-3	-4.71E-3	-1.14E-3	-3.37E-3	-5.31E-3	-7.10E-4	-2.66E-3	-4.86E-3	-4.81E-4	-2.36E-3	-4.75E-3
5	-5.74E-3	-3.90E-2	-1.21E-1	-1.39E-2	-4.83E-2	-1.28E-1	-1.08E-2	-4.31E-2	-1.23E-1	-8.82E-3	-4.06E-2	-1.21E-1
6	-1.51E-1	-5.61E-1	-8.66E-1	-2.40E-1	-6.13E-1	-8.71E-1	-2.08E-1	-5.84E-1	-8.67E-1	-1.87E-1	-5.70E-1	-8.66E-1

Table 2.121: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{7}$, and $q = 1$

k=7 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	6.04E-1	7.79E-1	9.24E-1	6.36E-1	7.93E-1	9.26E-1	6.27E-1	7.86E-1	9.24E-1	6.20E-1	7.82E-1	9.24E-1
1	-1.45E-12	-3.42E-10	-1.27E-9	-5.35E-8	-1.54E-8	-4.69E-9	-7.37E-9	-3.02E-9	-1.96E-9	-1.14E-9	-9.69E-10	-1.41E-9
2	-1.36E-9	-8.94E-8	-2.35E-7	-1.14E-6	-8.30E-7	-4.81E-7	-2.67E-7	-2.99E-7	-2.96E-7	-7.21E-8	-1.55E-7	-2.49E-7
3	-2.05E-7	-5.87E-6	-1.26E-5	-1.80E-5	-2.40E-5	-1.95E-5	-6.42E-6	-1.24E-5	-1.45E-5	-2.60E-6	-8.23E-6	-1.30E-5
4	-1.24E-5	-1.90E-4	-3.70E-4	-2.23E-4	-4.59E-4	-4.91E-4	-1.11E-4	-3.01E-4	-4.05E-4	-6.14E-5	-2.33E-4	-3.78E-4
5	-3.96E-4	-3.75E-3	-8.35E-3	-2.25E-3	-6.30E-3	-1.00E-2	-1.46E-3	-4.91E-3	-8.84E-3	-1.01E-3	-4.23E-3	-8.47E-3
6	-7.99E-3	-5.10E-2	-1.61E-1	-2.03E-2	-6.65E-2	-1.76E-1	-1.59E-2	-5.85E-2	-1.65E-1	-1.30E-2	-5.42E-2	-1.62E-1
7	-1.77E-1	-6.27E-1	-8.88E-1	-3.02E-1	-7.04E-1	-8.94E-1	-2.59E-1	-6.65E-1	-8.90E-1	-2.31E-1	-6.43E-1	-8.89E-1

Table 2.122: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{8}$, and $q = 1$

k=8 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	6.12E-1	7.91E-1	9.31E-1	6.48E-1	8.08E-1	9.35E-1	6.39E-1	8.00E-1	9.32E-1	6.31E-1	7.95E-1	9.31E-1
1	-8.88E-14	-3.18E-11	-1.30E-10	-3.05E-8	-6.04E-9	-1.14E-9	-3.76E-9	-8.82E-10	-3.11E-10	-4.99E-10	-1.94E-10	-1.68E-10
2	-8.45E-11	-8.35E-9	-2.42E-8	-4.73E-7	-2.22E-7	-8.52E-8	-9.23E-8	-5.97E-8	-3.91E-8	-2.03E-8	-2.31E-8	-2.77E-8
3	-1.32E-8	-5.55E-7	-1.29E-6	-5.71E-6	-5.03E-6	-2.91E-6	-1.66E-6	-2.01E-6	-1.75E-6	-5.45E-7	-1.07E-6	-1.40E-6
4	-8.45E-7	-1.85E-5	-3.68E-5	-5.79E-5	-8.20E-5	-6.37E-5	-2.36E-5	-4.36E-5	-4.51E-5	-1.06E-5	-2.84E-5	-3.90E-5
5	-2.97E-5	-3.88E-4	-7.43E-4	-4.99E-4	-1.03E-3	-1.08E-3	-2.68E-4	-6.76E-4	-8.52E-4	-1.56E-4	-5.12E-4	-7.72E-4
6	-6.59E-4	-5.69E-3	-1.31E-2	-3.74E-3	-1.03E-2	-1.68E-2	-2.52E-3	-7.95E-3	-1.44E-2	-1.80E-3	-6.72E-3	-1.35E-2
7	-1.04E-2	-6.31E-2	-2.00E-1	-2.75E-2	-8.63E-2	-2.24E-1	-2.17E-2	-7.52E-2	-2.08E-1	-1.79E-2	-6.88E-2	-2.02E-1
8	-2.03E-1	-6.88E-1	-9.02E-1	-3.69E-1	-7.95E-1	-9.07E-1	-3.15E-1	-7.45E-1	-9.04E-1	-2.78E-1	-7.15E-1	-9.02E-1

Table 2.123: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{3}$, and $q = 100$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	9.92E-3	9.96E-3	9.98E-3	9.92E-3	9.96E-3	9.98E-3	9.92E-3	9.96E-3	9.98E-3	9.92E-3	9.96E-3	9.98E-3
1	-8.88E-9	-1.13E-6	-1.17E-5	-3.94E-8	-1.27E-6	-1.18E-5	-2.26E-8	-1.17E-6	-1.17E-5	-1.55E-8	-1.14E-6	-1.17E-5
2	-7.98E-7	-1.10E-5	-3.24E-5	-1.36E-6	-1.12E-5	-3.23E-5	-1.10E-6	-1.10E-5	-3.24E-5	-9.62E-7	-1.10E-5	-3.24E-5
3	-2.16E-5	-5.53E-5	-5.19E-5	-2.59E-5	-5.58E-5	-5.19E-5	-2.40E-5	-5.55E-5	-5.19E-5	-2.29E-5	-5.54E-5	-5.19E-5

Table 2.124: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{4}$, and $q = 100$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	9.93E-3	9.96E-3	9.99E-3	9.93E-3	9.96E-3	9.99E-3	9.93E-3	9.96E-3	9.99E-3	9.93E-3	9.96E-3	9.99E-3
1	-6.77E-10	-2.10E-7	-2.13E-6	-1.13E-8	-3.15E-7	-2.30E-6	-4.71E-9	-2.44E-7	-2.16E-6	-2.37E-9	-2.21E-7	-2.13E-6
2	-6.90E-8	-3.26E-6	-2.20E-5	-2.39E-7	-3.63E-6	-2.21E-5	-1.56E-7	-3.39E-6	-2.21E-5	-1.14E-7	-3.31E-6	-2.20E-5
3	-1.71E-6	-1.46E-5	-2.93E-5	-3.00E-6	-1.52E-5	-2.92E-5	-2.45E-6	-1.48E-5	-2.92E-5	-2.13E-6	-1.47E-5	-2.93E-5
4	-2.82E-5	-6.25E-5	-5.15E-5	-3.58E-5	-6.37E-5	-5.15E-5	-3.27E-5	-6.29E-5	-5.15E-5	-3.08E-5	-6.26E-5	-5.15E-5

Table 2.125: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{5}$, and $q = 100$

k=5 /δ j /α	0.00 0.25	0.00 0.50	0.00 0.75	10^{-6} 0.25	10^{-6} 0.50	10^{-6} 0.75	10^{-7} 0.25	10^{-7} 0.50	10^{-7} 0.75	10^{-8} 0.25	10^{-8} 0.50	10^{-8} 0.75
0	9.93E-3	9.97E-3	9.99E-3	9.94E-3	9.97E-3	9.99E-3	9.93E-3	9.97E-3	9.99E-3	9.93E-3	9.97E-3	9.99E-3
1	-5.91E-11	-2.82E-8	-1.79E-7	-4.95E-9	-8.32E-8	-2.36E-7	-1.62E-9	-4.53E-8	-1.91E-7	-6.06E-10	-3.36E-8	-1.81E-7
2	-7.40E-9	-9.46E-7	-9.98E-6	-6.77E-8	-1.31E-6	-1.06E-5	-3.57E-8	-1.09E-6	-1.01E-5	-2.12E-8	-9.95E-7	-1.00E-5
3	-1.94E-7	-5.08E-6	-2.24E-5	-6.42E-7	-5.73E-6	-2.22E-5	-4.45E-7	-5.34E-6	-2.23E-5	-3.34E-7	-5.17E-6	-2.24E-5
4	-2.71E-6	-1.73E-5	-2.78E-5	-4.93E-6	-1.82E-5	-2.77E-5	-4.08E-6	-1.77E-5	-2.78E-5	-3.53E-6	-1.74E-5	-2.78E-5
5	-3.42E-5	-6.87E-5	-5.19E-5	-4.60E-5	-7.10E-5	-5.19E-5	-4.15E-5	-6.96E-5	-5.19E-5	-3.86E-5	-6.90E-5	-5.19E-5

Table 2.126: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{6}$, and $q = 100$

k=6 /δ j /α	0.00 0.25	0.00 0.50	0.00 0.75	10^{-6} 0.25	10^{-6} 0.50	10^{-6} 0.75	10^{-7} 0.25	10^{-7} 0.50	10^{-7} 0.75	10^{-8} 0.25	10^{-8} 0.50	10^{-8} 0.75
0	9.93E-3	9.97E-3	9.99E-3	9.94E-3	9.97E-3	9.99E-3	9.94E-3	9.97E-3	9.99E-3	9.94E-3	9.97E-3	9.99E-3
1	-5.37E-12	-2.95E-9	-1.40E-8	-2.75E-9	-2.57E-8	-2.74E-8	-7.58E-10	-8.96E-9	-1.68E-8	-2.26E-10	-4.70E-9	-1.46E-8
2	-8.93E-10	-2.24E-7	-2.02E-6	-2.64E-8	-5.04E-7	-2.69E-6	-1.16E-8	-3.35E-7	-2.18E-6	-5.65E-9	-2.65E-7	-2.05E-6
3	-2.60E-8	-1.84E-6	-1.60E-5	-1.96E-7	-2.49E-6	-1.65E-5	-1.15E-7	-2.13E-6	-1.61E-5	-7.40E-8	-1.95E-6	-1.60E-5
4	-3.73E-7	-6.58E-6	-2.08E-5	-1.22E-6	-7.57E-6	-2.06E-5	-8.83E-7	-7.03E-6	-2.08E-5	-6.77E-7	-6.75E-6	-2.08E-5
5	-3.73E-6	-1.95E-5	-2.71E-5	-7.00E-6	-2.09E-5	-2.70E-5	-5.85E-6	-2.01E-5	-2.70E-5	-5.07E-6	-1.97E-5	-2.71E-5
6	-3.97E-5	-7.42E-5	-5.25E-5	-5.66E-5	-7.82E-5	-5.27E-5	-5.07E-5	-7.60E-5	-5.26E-5	-4.67E-5	-7.49E-5	-5.25E-5

Table 2.127: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{7}$, and $q = 100$

k=7 /δ j /α	0.00 0.25	0.00 0.50	0.00 0.75	10^{-6} 0.25	10^{-6} 0.50	10^{-6} 0.75	10^{-7} 0.25	10^{-7} 0.50	10^{-7} 0.75	10^{-8} 0.25	10^{-8} 0.50	10^{-8} 0.75
0	9.93E-3	9.97E-3	9.99E-3	9.94E-3	9.97E-3	9.99E-3	9.94E-3	9.97E-3	9.99E-3	9.94E-3	9.97E-3	9.99E-3
1	-4.89E-13	-2.88E-10	-1.25E-9	-1.76E-9	-9.71E-9	-4.64E-9	-4.31E-10	-2.24E-9	-1.94E-9	-1.09E-10	-7.75E-10	-1.40E-9
2	-1.14E-10	-3.98E-8	-2.25E-7	-1.28E-8	-2.02E-7	-4.56E-7	-4.84E-9	-9.99E-8	-2.82E-7	-1.98E-9	-6.13E-8	-2.38E-7
3	-3.86E-9	-5.97E-7	-6.42E-6	-7.72E-8	-1.17E-6	-8.25E-6	-3.92E-8	-8.68E-7	-6.98E-6	-2.16E-8	-7.10E-7	-6.55E-6
4	-5.89E-8	-2.67E-6	-1.74E-5	-4.05E-7	-3.62E-6	-1.73E-5	-2.56E-7	-3.14E-6	-1.74E-5	-1.72E-7	-2.88E-6	-1.74E-5
5	-5.92E-7	-7.84E-6	-1.98E-5	-1.93E-6	-9.20E-6	-1.95E-5	-1.44E-6	-8.52E-6	-1.97E-5	-1.13E-6	-8.14E-6	-1.97E-5
6	-4.73E-6	-2.13E-5	-2.66E-5	-9.12E-6	-2.32E-5	-2.65E-5	-7.69E-6	-2.22E-5	-2.66E-5	-6.69E-6	-2.17E-5	-2.66E-5
7	-4.49E-5	-7.94E-5	-5.33E-5	-6.78E-5	-8.54E-5	-5.36E-5	-6.02E-5	-8.23E-5	-5.34E-5	-5.50E-5	-8.06E-5	-5.33E-5

Table 2.128: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{8}$, and $q = 100$

k=8 /δ j /α	0.00 0.25	0.00 0.50	0.00 0.75	10^{-6} 0.25	10^{-6} 0.50	10^{-6} 0.75	10^{-7} 0.25	10^{-7} 0.50	10^{-7} 0.75	10^{-8} 0.25	10^{-8} 0.50	10^{-8} 0.75
0	9.94E-3	9.97E-3	9.99E-3	9.95E-3	9.98E-3	9.99E-3	9.94E-3	9.98E-3	9.99E-3	9.94E-3	9.97E-3	9.99E-3
1	-4.45E-14	-2.94E-11	-1.30E-10	-1.23E-9	-4.38E-9	-1.13E-9	-2.77E-10	-7.31E-10	-3.10E-10	-6.17E-11	-1.71E-10	-1.67E-10
2	-1.45E-11	-5.66E-9	-2.40E-8	-7.22E-9	-8.58E-8	-8.53E-8	-2.42E-9	-3.08E-8	-3.89E-8	-8.56E-10	-1.39E-8	-2.75E-8
3	-6.04E-10	-1.54E-7	-1.18E-6	-3.64E-8	-5.70E-7	-2.47E-6	-1.63E-8	-3.48E-7	-1.57E-6	-7.80E-9	-2.37E-7	-1.28E-6
4	-1.01E-8	-1.04E-6	-1.09E-5	-1.65E-7	-1.88E-6	-1.28E-5	-9.20E-8	-1.49E-6	-1.17E-5	-5.46E-8	-1.25E-6	-1.12E-5
5	-1.06E-7	-3.42E-6	-1.68E-5	-6.88E-7	-4.67E-6	-1.63E-5	-4.57E-7	-4.10E-6	-1.66E-5	-3.20E-7	-3.75E-6	-1.68E-5
6	-8.39E-7	-8.92E-6	-1.91E-5	-2.72E-6	-1.06E-5	-1.88E-5	-2.09E-6	-9.87E-6	-1.90E-5	-1.66E-6	-9.38E-6	-1.91E-5
7	-5.69E-6	-2.28E-5	-2.64E-5	-1.13E-5	-2.54E-5	-2.63E-5	-9.56E-6	-2.42E-5	-2.64E-5	-8.34E-6	-2.35E-5	-2.64E-5
8	-4.99E-5	-8.41E-5	-5.41E-5	-7.95E-5	-9.27E-5	-5.47E-5	-7.01E-5	-8.87E-5	-5.43E-5	-6.36E-5	-8.63E-5	-5.42E-5

Table 2.129: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 3}$, and $q = 200$

k=3 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3
1	-2.41E-9	-3.29E-7	-3.66E-6	-1.05E-8	-3.66E-7	-3.69E-6	-6.04E-9	-3.41E-7	-3.67E-6	-4.18E-9	-3.33E-7	-3.66E-6
2	-2.04E-7	-2.77E-6	-7.55E-6	-3.46E-7	-2.84E-6	-7.54E-6	-2.81E-7	-2.79E-6	-7.55E-6	-2.46E-7	-2.78E-6	-7.55E-6
3	-5.43E-6	-1.38E-5	-1.28E-5	-6.51E-6	-1.40E-5	-1.28E-5	-6.04E-6	-1.39E-5	-1.28E-5	-5.78E-6	-1.38E-5	-1.28E-5

Table 2.130: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 4}$, and $q = 200$

k=4 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3
1	-2.00E-10	-7.79E-8	-1.15E-6	-3.10E-9	-1.11E-7	-1.22E-6	-1.32E-9	-8.89E-8	-1.17E-6	-6.76E-10	-8.15E-8	-1.16E-6
2	-1.82E-8	-8.62E-7	-5.41E-6	-6.19E-8	-9.49E-7	-5.41E-6	-4.06E-8	-8.93E-7	-5.41E-6	-2.99E-8	-8.72E-7	-5.41E-6
3	-4.35E-7	-3.66E-6	-6.95E-6	-7.61E-7	-3.80E-6	-6.94E-6	-6.23E-7	-3.71E-6	-6.95E-6	-5.42E-7	-3.68E-6	-6.95E-6
4	-7.10E-6	-1.56E-5	-1.28E-5	-8.99E-6	-1.59E-5	-1.28E-5	-8.22E-6	-1.57E-5	-1.28E-5	-7.75E-6	-1.56E-5	-1.28E-5

Table 2.131: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 5}$, and $q = 200$

k=5 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3
1	-2.00E-11	-1.49E-8	-1.52E-7	-1.40E-9	-3.77E-8	-1.97E-7	-4.74E-10	-2.24E-8	-1.62E-7	-1.85E-10	-1.73E-8	-1.54E-7
2	-2.04E-9	-2.84E-7	-3.34E-6	-1.78E-8	-3.75E-7	-3.46E-6	-9.51E-9	-3.20E-7	-3.37E-6	-5.72E-9	-2.96E-7	-3.35E-6
3	-5.03E-8	-1.30E-6	-5.02E-6	-1.65E-7	-1.46E-6	-4.99E-6	-1.14E-7	-1.36E-6	-5.01E-6	-8.63E-8	-1.32E-6	-5.02E-6
4	-6.89E-7	-4.33E-6	-6.70E-6	-1.25E-6	-4.57E-6	-6.69E-6	-1.03E-6	-4.43E-6	-6.70E-6	-8.95E-7	-4.37E-6	-6.70E-6
5	-8.60E-6	-1.72E-5	-1.29E-5	-1.16E-5	-1.77E-5	-1.29E-5	-1.04E-5	-1.74E-5	-1.29E-5	-9.72E-6	-1.72E-5	-1.29E-5

Table 2.132: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 6}$, and $q = 200$

k=6 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3
1	-2.17E-12	-2.10E-9	-1.34E-8	-7.98E-10	-1.45E-8	-2.61E-8	-2.32E-10	-5.75E-9	-1.61E-8	-7.35E-11	-3.21E-9	-1.40E-8
2	-2.67E-10	-8.62E-8	-1.21E-6	-7.08E-9	-1.66E-7	-1.48E-6	-3.17E-9	-1.20E-7	-1.28E-6	-1.58E-9	-9.86E-8	-1.22E-6
3	-6.97E-9	-4.99E-7	-4.05E-6	-5.09E-8	-6.54E-7	-4.04E-6	-3.02E-8	-5.68E-7	-4.05E-6	-1.95E-8	-5.26E-7	-4.05E-6
4	-9.62E-8	-1.66E-6	-4.73E-6	-3.12E-7	-1.91E-6	-4.70E-6	-2.26E-7	-1.77E-6	-4.72E-6	-1.74E-7	-1.70E-6	-4.73E-6
5	-9.46E-7	-4.87E-6	-6.58E-6	-1.77E-6	-5.22E-6	-6.56E-6	-1.48E-6	-5.03E-6	-6.57E-6	-1.28E-6	-4.93E-6	-6.58E-6
6	-9.98E-6	-1.85E-5	-1.30E-5	-1.42E-5	-1.95E-5	-1.31E-5	-1.27E-5	-1.90E-5	-1.31E-5	-1.17E-5	-1.87E-5	-1.31E-5

Table 2.133: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{7}$, and $q = 200$

k=7 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.98E-3	4.99E-3	5.00E-3	4.99E-3	4.99E-3	5.00E-3	4.99E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3
1	-2.41E-13	-2.44E-10	-1.24E-9	-5.22E-10	-6.42E-9	-4.57E-9	-1.37E-10	-1.70E-9	-1.91E-9	-3.74E-11	-6.28E-10	-1.38E-9
2	-3.82E-11	-2.11E-8	-2.01E-7	-3.49E-9	-7.95E-8	-3.83E-7	-1.36E-9	-4.53E-8	-2.48E-7	-5.77E-10	-3.03E-8	-2.11E-7
3	-1.09E-9	-1.90E-7	-2.55E-6	-2.03E-8	-3.29E-7	-2.88E-6	-1.04E-8	-2.57E-7	-2.66E-6	-5.81E-9	-2.19E-7	-2.57E-6
4	-1.55E-8	-6.97E-7	-3.88E-6	-1.04E-7	-9.24E-7	-3.79E-6	-6.61E-8	-8.09E-7	-3.85E-6	-4.47E-8	-7.45E-7	-3.87E-6
5	-1.52E-7	-1.97E-6	-4.58E-6	-4.91E-7	-2.31E-6	-4.55E-6	-3.67E-7	-2.14E-6	-4.57E-6	-2.88E-7	-2.04E-6	-4.58E-6
6	-1.20E-6	-5.32E-6	-6.51E-6	-2.30E-6	-5.81E-6	-6.50E-6	-1.94E-6	-5.56E-6	-6.51E-6	-1.69E-6	-5.42E-6	-6.51E-6
7	-1.13E-5	-1.98E-5	-1.33E-5	-1.70E-5	-2.13E-5	-1.33E-5	-1.51E-5	-2.06E-5	-1.33E-5	-1.38E-5	-2.01E-5	-1.33E-5

Table 2.134: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{8}$, and $q = 200$

k=8 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.98E-3	4.99E-3	5.00E-3	4.99E-3	4.99E-3	5.00E-3	4.99E-3	4.99E-3	5.00E-3	4.99E-3	4.99E-3	5.00E-3
1	-2.65E-14	-2.72E-11	-1.29E-10	-3.73E-10	-3.22E-9	-1.13E-9	-9.09E-11	-6.08E-10	-3.08E-10	-2.23E-11	-1.52E-10	-1.66E-10
2	-5.69E-12	-3.96E-9	-2.35E-8	-2.00E-9	-4.04E-8	-8.25E-8	-6.98E-10	-1.74E-8	-3.81E-8	-2.59E-10	-8.79E-9	-2.70E-8
3	-1.86E-10	-6.42E-8	-8.48E-7	-9.69E-9	-1.79E-7	-1.44E-6	-4.41E-9	-1.22E-7	-1.05E-6	-2.16E-9	-9.01E-8	-9.02E-7
4	-2.79E-9	-2.99E-7	-3.19E-6	-4.29E-8	-4.96E-7	-3.25E-6	-2.41E-8	-4.04E-7	-3.23E-6	-1.44E-8	-3.49E-7	-3.21E-6
5	-2.78E-8	-8.77E-7	-3.67E-6	-1.76E-7	-1.18E-6	-3.58E-6	-1.17E-7	-1.04E-6	-3.63E-6	-8.24E-8	-9.56E-7	-3.66E-6
6	-2.15E-7	-2.24E-6	-4.50E-6	-6.91E-7	-2.67E-6	-4.46E-6	-5.30E-7	-2.47E-6	-4.48E-6	-4.22E-7	-2.35E-6	-4.49E-6
7	-1.44E-6	-5.71E-6	-6.48E-6	-2.85E-6	-6.35E-6	-6.47E-6	-2.41E-6	-6.05E-6	-6.47E-6	-2.11E-6	-5.87E-6	-6.47E-6
8	-1.25E-5	-2.10E-5	-1.35E-5	-2.00E-5	-2.31E-5	-1.36E-5	-1.76E-5	-2.21E-5	-1.35E-5	-1.60E-5	-2.16E-5	-1.35E-5

Table 2.135: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{3}$, and $q = 400$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.49E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3
1	-6.27E-10	-8.91E-8	-1.02E-6	-2.70E-9	-9.86E-8	-1.03E-6	-1.56E-9	-9.22E-8	-1.02E-6	-1.08E-9	-9.01E-8	-1.02E-6
2	-5.17E-8	-6.96E-7	-1.80E-6	-8.75E-8	-7.13E-7	-1.80E-6	-7.10E-8	-7.01E-7	-1.80E-6	-6.22E-8	-6.97E-7	-1.80E-6
3	-1.36E-6	-3.46E-6	-3.18E-6	-1.63E-6	-3.49E-6	-3.18E-6	-1.52E-6	-3.47E-6	-3.18E-6	-1.45E-6	-3.46E-6	-3.18E-6

Table 2.136: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{4}$, and $q = 400$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3
1	-5.46E-11	-2.44E-8	-4.53E-7	-8.14E-10	-3.35E-8	-4.72E-7	-3.50E-10	-2.75E-8	-4.57E-7	-1.81E-10	-2.54E-8	-4.54E-7
2	-4.66E-9	-2.20E-7	-1.25E-6	-1.58E-8	-2.41E-7	-1.25E-6	-1.04E-8	-2.28E-7	-1.25E-6	-7.64E-9	-2.23E-7	-1.25E-6
3	-1.10E-7	-9.18E-7	-1.69E-6	-1.92E-7	-9.52E-7	-1.69E-6	-1.57E-7	-9.30E-7	-1.69E-6	-1.37E-7	-9.22E-7	-1.69E-6
4	-1.78E-6	-3.90E-6	-3.17E-6	-2.26E-6	-3.98E-6	-3.17E-6	-2.06E-6	-3.93E-6	-3.17E-6	-1.94E-6	-3.91E-6	-3.17E-6

Table 2.137: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{5}$, and $q = 400$

k=5 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3
1	-5.91E-12	-6.10E-9	-1.06E-7	-3.73E-10	-1.36E-8	-1.31E-7	-1.29E-10	-8.68E-9	-1.11E-7	-5.13E-11	-6.95E-9	-1.07E-7
2	-5.38E-10	-7.71E-8	-9.12E-7	-4.58E-9	-9.89E-8	-9.23E-7	-2.46E-9	-8.56E-8	-9.15E-7	-1.49E-9	-8.01E-8	-9.12E-7
3	-1.28E-8	-3.27E-7	-1.16E-6	-4.17E-8	-3.66E-7	-1.15E-6	-2.90E-8	-3.42E-7	-1.16E-6	-2.19E-8	-3.32E-7	-1.16E-6
4	-1.74E-7	-1.08E-6	-1.64E-6	-3.14E-7	-1.14E-6	-1.64E-6	-2.60E-7	-1.11E-6	-1.64E-6	-2.25E-7	-1.09E-6	-1.64E-6
5	-2.16E-6	-4.29E-6	-3.21E-6	-2.90E-6	-4.43E-6	-3.21E-6	-2.62E-6	-4.34E-6	-3.21E-6	-2.44E-6	-4.31E-6	-3.21E-6

Table 2.138: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{6}$, and $q = 400$

k=6 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3
1	-7.26E-13	-1.20E-9	-1.22E-8	-2.16E-10	-6.32E-9	-2.31E-8	-6.47E-11	-2.88E-9	-1.46E-8	-2.12E-11	-1.74E-9	-1.27E-8
2	-7.36E-11	-2.71E-8	-4.91E-7	-1.84E-9	-4.70E-8	-5.56E-7	-8.31E-10	-3.57E-8	-5.09E-7	-4.19E-10	-3.04E-8	-4.95E-7
3	-1.81E-9	-1.29E-7	-9.15E-7	-1.30E-8	-1.66E-7	-9.04E-7	-7.71E-9	-1.45E-7	-9.12E-7	-5.00E-9	-1.35E-7	-9.15E-7
4	-2.44E-8	-4.17E-7	-1.12E-6	-7.87E-8	-4.78E-7	-1.12E-6	-5.71E-8	-4.44E-7	-1.12E-6	-4.39E-8	-4.28E-7	-1.12E-6
5	-2.38E-7	-1.22E-6	-1.62E-6	-4.45E-7	-1.31E-6	-1.62E-6	-3.72E-7	-1.26E-6	-1.62E-6	-3.23E-7	-1.23E-6	-1.62E-6
6	-2.50E-6	-4.63E-6	-3.25E-6	-3.57E-6	-4.88E-6	-3.26E-6	-3.19E-6	-4.74E-6	-3.25E-6	-2.94E-6	-4.68E-6	-3.25E-6

Table 2.139: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{7}$, and $q = 400$

k=7 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3
1	-9.46E-14	-1.81E-10	-1.20E-9	-1.43E-10	-3.30E-9	-4.40E-9	-3.90E-11	-1.05E-9	-1.86E-9	-1.12E-11	-4.31E-10	-1.34E-9
2	-1.13E-11	-8.55E-9	-1.44E-7	-9.13E-10	-2.50E-8	-2.36E-7	-3.62E-10	-1.59E-8	-1.70E-7	-1.56E-10	-1.15E-8	-1.50E-7
3	-2.91E-10	-5.28E-8	-7.25E-7	-5.20E-9	-8.57E-8	-7.45E-7	-2.69E-9	-6.87E-8	-7.34E-7	-1.51E-9	-5.96E-8	-7.28E-7
4	-4.00E-9	-1.77E-7	-8.63E-7	-2.64E-8	-2.33E-7	-8.51E-7	-1.68E-8	-2.04E-7	-8.59E-7	-1.14E-8	-1.89E-7	-8.62E-7
5	-3.85E-8	-4.94E-7	-1.10E-6	-1.24E-7	-5.78E-7	-1.10E-6	-9.27E-8	-5.36E-7	-1.10E-6	-7.26E-8	-5.12E-7	-1.10E-6
6	-3.01E-7	-1.33E-6	-1.61E-6	-5.79E-7	-1.45E-6	-1.61E-6	-4.88E-7	-1.39E-6	-1.61E-6	-4.25E-7	-1.36E-6	-1.61E-6
7	-2.83E-6	-4.95E-6	-3.30E-6	-4.27E-6	-5.33E-6	-3.32E-6	-3.79E-6	-5.14E-6	-3.31E-6	-3.46E-6	-5.03E-6	-3.30E-6

Table 2.140: The coefficients $\bar{c}_j, j = 0, \ldots, k$ of the partial fraction representation (1.47) of (q, δ, α, k)-0-URA approximation (1.46) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{8}$, and $q = 400$

k=8 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3
1	-1.25E-14	-2.34E-11	-1.28E-10	-1.03E-10	-1.88E-9	-1.11E-9	-2.64E-11	-4.33E-10	-3.05E-10	-6.88E-12	-1.20E-10	-1.65E-10
2	-1.86E-12	-2.18E-9	-2.20E-8	-5.28E-10	-1.44E-8	-7.09E-8	-1.88E-10	-7.40E-9	-3.48E-8	-7.16E-11	-4.25E-9	-2.51E-8
3	-5.18E-11	-2.10E-8	-4.00E-7	-2.50E-9	-4.87E-8	-5.32E-7	-1.15E-9	-3.52E-8	-4.53E-7	-5.68E-10	-2.76E-8	-4.15E-7
4	-7.31E-10	-7.85E-8	-7.37E-7	-1.09E-8	-1.26E-7	-7.13E-7	-6.16E-9	-1.04E-7	-7.29E-7	-3.70E-9	-9.04E-8	-7.35E-7
5	-7.10E-9	-2.21E-7	-8.36E-7	-4.46E-8	-2.96E-7	-8.26E-7	-2.98E-8	-2.61E-7	-8.32E-7	-2.09E-8	-2.40E-7	-8.35E-7
6	-5.43E-8	-5.60E-7	-1.09E-6	-1.74E-7	-6.67E-7	-1.08E-6	-1.34E-7	-6.19E-7	-1.09E-6	-1.07E-7	-5.89E-7	-1.09E-6
7	-3.62E-7	-1.43E-6	-1.60E-6	-7.15E-7	-1.59E-6	-1.60E-6	-6.06E-7	-1.51E-6	-1.60E-6	-5.29E-7	-1.47E-6	-1.60E-6
8	-3.14E-6	-5.25E-6	-3.36E-6	-5.00E-6	-5.78E-6	-3.39E-6	-4.41E-6	-5.53E-6	-3.37E-6	-4.01E-6	-5.39E-6	-3.36E-6

2.7 Tables type (g) for 1-URA-poles

Table 2.141: The poles of $\bar{\bar{r}}_{q,\delta,\alpha,k}(t)$, (1.48), of (q, δ, α, k)-1-URA approximation, i.e. the rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 3}$, and $q_0 = q_1 = 100$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.73E-14	-2.35E-9	-1.18E-7	-1.21E-9	-1.92E-8	-1.64E-7	-2.51E-10	-7.07E-9	-1.28E-7	-5.17E-11	-3.76E-9	-1.20E-7
2	-1.15E-11	-1.41E-7	-3.74E-6	-2.68E-8	-3.10E-7	-3.93E-6	-7.54E-9	-2.08E-7	-3.79E-6	-2.12E-9	-1.65E-7	-3.75E-6
3	-3.78E-9	-2.02E-6	-8.77E-6	-1.19E-6	-3.24E-6	-8.85E-6	-4.97E-7	-2.53E-6	-8.79E-6	-1.94E-7	-2.21E-6	-8.78E-6

Table 2.142: The poles of $\bar{\bar{r}}_{q,\delta,\alpha,k}(t)$, (1.48), of (q, δ, α, k)-1-URA approximation, i.e. the rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 4}$, and $q_0 = q_1 = 100$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.23E-15	-2.31E-10	-8.64E-9	-7.02E-10	-7.79E-9	-1.94E-8	-1.42E-10	-1.88E-9	-1.09E-8	-2.77E-11	-6.48E-10	-9.11E-9
2	-7.73E-13	-2.67E-8	-1.14E-6	-8.06E-9	-1.19E-7	-1.49E-6	-2.16E-9	-6.32E-8	-1.23E-6	-5.73E-10	-4.03E-8	-1.16E-6
3	-9.92E-11	-3.57E-7	-4.87E-6	-1.21E-7	-7.57E-7	-4.94E-6	-4.35E-8	-5.37E-7	-4.89E-6	-1.49E-8	-4.31E-7	-4.88E-6
4	-1.80E-8	-3.55E-6	-9.58E-6	-3.00E-6	-5.82E-6	-9.74E-6	-1.62E-6	-4.65E-6	-9.62E-6	-8.05E-7	-4.02E-6	-9.59E-6

Table 2.143: The poles of $\bar{\bar{r}}_{q,\delta,\alpha,k}(t)$, (1.48), of (q, δ, α, k)-1-URA approximation, i.e. the rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 5}$, and $q_0 = q_1 = 100$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.07E-16	-2.38E-11	-7.55E-10	-4.69E-10	-3.72E-9	-3.42E-9	-9.24E-11	-6.43E-10	-1.28E-9	-1.74E-11	-1.49E-10	-8.64E-10
2	-7.29E-14	-4.26E-9	-1.52E-7	-3.56E-9	-5.39E-8	-3.38E-7	-9.16E-10	-2.17E-8	-1.99E-7	-2.31E-10	-1.03E-8	-1.63E-7
3	-7.66E-12	-8.99E-8	-2.66E-6	-3.11E-8	-3.02E-7	-3.14E-6	-1.02E-8	-1.90E-7	-2.83E-6	-3.28E-9	-1.34E-7	-2.70E-6
4	-4.91E-10	-6.47E-7	-5.02E-6	-3.31E-7	-1.37E-6	-5.05E-6	-1.46E-7	-1.01E-6	-5.03E-6	-6.06E-8	-8.12E-7	-5.02E-6
5	-6.21E-8	-5.25E-6	-1.02E-5	-5.23E-6	-8.47E-6	-1.04E-5	-3.33E-6	-7.00E-6	-1.03E-5	-1.99E-6	-6.09E-6	-1.02E-5

Table 2.144: The poles of $\bar{\bar{r}}_{q,\delta,\alpha,k}(t)$, (1.48), of (q, δ, α, k)-1-URA approximation, i.e. the rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k = 6}$, and $q_0 = q_1 = 100$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.07E-17	-2.67E-12	-7.85E-11	-3.40E-10	-2.01E-9	-8.77E-10	-6.57E-11	-2.71E-10	-2.15E-10	-1.20E-11	-4.57E-11	-1.07E-10
2	-8.01E-15	-5.91E-10	-1.57E-8	-1.95E-9	-2.70E-8	-6.76E-8	-4.83E-10	-8.16E-9	-2.80E-8	-1.16E-10	-2.83E-9	-1.86E-8
3	-8.51E-13	-2.06E-8	-7.47E-7	-1.20E-8	-1.48E-7	-1.45E-6	-3.73E-9	-7.90E-8	-9.88E-7	-1.13E-9	-4.59E-8	-8.13E-7
4	-4.02E-11	-1.85E-7	-3.54E-6	-8.25E-8	-5.51E-7	-3.72E-6	-3.23E-8	-3.74E-7	-3.64E-6	-1.22E-8	-2.77E-7	-3.57E-6
5	-1.75E-9	-9.93E-7	-5.10E-6	-6.58E-7	-2.10E-6	-5.18E-6	-3.41E-7	-1.60E-6	-5.12E-6	-1.66E-7	-1.30E-6	-5.10E-6
6	-1.67E-7	-6.93E-6	-1.07E-5	-7.62E-6	-1.09E-5	-1.10E-5	-5.31E-6	-9.26E-6	-1.08E-5	-3.58E-6	-8.17E-6	-1.07E-5

Table 2.145: The poles of $\bar{\bar{r}}_{q,\delta,\alpha,k}(t)$, (1.48), of (q, δ, α, k)-1-URA approximation, i.e. the rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{7}$, and $q_0 = q_1 = 100$

k=7 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-1.19E-18	-3.29E-13	-9.39E-12	-2.61E-10	-1.19E-9	-3.00E-10	-4.95E-11	-1.34E-10	-5.06E-11	-8.77E-12	-1.77E-11	-1.72E-11
2	-9.61E-16	-8.00E-11	-1.81E-9	-1.22E-9	-1.46E-8	-1.65E-8	-2.92E-10	-3.39E-9	-4.95E-9	-6.74E-11	-8.79E-10	-2.57E-9
3	-1.10E-13	-3.90E-9	-1.07E-7	-5.90E-9	-8.05E-8	-4.55E-7	-1.73E-9	-3.51E-8	-2.08E-7	-4.97E-10	-1.61E-8	-1.35E-7
4	-4.90E-12	-5.41E-8	-1.72E-6	-3.04E-8	-2.77E-7	-2.61E-6	-1.10E-8	-1.71E-7	-2.14E-6	-3.84E-9	-1.12E-7	-1.87E-6
5	-1.47E-10	-3.03E-7	-3.74E-6	-1.70E-7	-8.53E-7	-3.70E-6	-7.68E-8	-6.06E-7	-3.73E-6	-3.31E-8	-4.62E-7	-3.74E-6
6	-4.96E-9	-1.38E-6	-5.21E-6	-1.07E-6	-2.85E-6	-5.35E-6	-6.26E-7	-2.24E-6	-5.27E-6	-3.47E-7	-1.85E-6	-5.23E-6
7	-3.66E-7	-8.48E-6	-1.11E-5	-1.01E-5	-1.31E-5	-1.14E-5	-7.41E-6	-1.13E-5	-1.12E-5	-5.36E-6	-1.01E-5	-1.11E-5

Table 2.146: The poles of $\bar{\bar{r}}_{q,\delta,\alpha,k}(t)$, (1.48), of (q, δ, α, k)-1-URA approximation, i.e. the rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{8}$, and $q_0 = q_1 = 100$

k=8 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	7.72E-20	-4.44E-14	-1.26E-12	-2.08E-10	-7.65E-10	-1.27E-10	-3.89E-11	-7.45E-11	-1.57E-11	-6.74E-12	-8.18E-12	-3.67E-12
2	1.80E-17	-1.12E-11	-2.38E-10	-8.36E-10	-8.45E-9	-5.05E-9	-1.94E-10	-1.56E-9	-1.14E-9	-4.31E-11	-3.16E-10	-4.46E-10
3	1.09E-20	-6.50E-10	-1.37E-8	-3.36E-9	-4.65E-8	-1.25E-7	-9.45E-10	-1.63E-8	-4.13E-8	-2.59E-10	-5.78E-9	-2.11E-8
4	-5.72E-27	-1.34E-8	-4.03E-7	-1.41E-8	-1.57E-7	-1.37E-6	-4.78E-9	-8.53E-8	-7.86E-7	-1.57E-9	-4.75E-8	-5.30E-7
5	1.50E-13	-1.02E-7	-2.55E-6	-6.23E-8	-4.34E-7	-3.02E-6	-2.56E-8	-2.88E-7	-2.87E-6	-1.01E-8	-2.02E-7	-2.69E-6
6	1.27E-25	-4.36E-7	-3.71E-6	-2.94E-7	-1.19E-6	-3.67E-6	-1.49E-7	-8.76E-7	-3.68E-6	-7.20E-8	-6.81E-7	-3.70E-6
7	3.58E-25	-1.78E-6	-5.34E-6	-1.55E-6	-3.59E-6	-5.52E-6	-9.82E-7	-2.91E-6	-5.42E-6	-5.99E-7	-2.44E-6	-5.37E-6
8	7.10E-24	-9.87E-6	-1.13E-5	-1.28E-5	-1.51E-5	-1.17E-5	-9.60E-6	-1.32E-5	-1.15E-5	-7.23E-6	-1.19E-5	-1.14E-5

Table 2.147: The poles of $\bar{\bar{r}}_{q,\delta,\alpha,k}(t)$, (1.48), of (q, δ, α, k)-1-URA approximation, i.e. the rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{3}$, and $q_0 = q_1 = 200$

k=3 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-5.88E-16	-3.33E-10	-2.80E-8	-3.03E-10	-5.31E-9	-4.72E-8	-6.15E-11	-1.70E-9	-3.22E-8	-1.22E-11	-7.25E-10	-2.89E-8
2	-3.94E-13	-1.92E-8	-7.86E-7	-6.25E-9	-5.92E-8	-8.46E-7	-1.63E-9	-3.54E-8	-8.02E-7	-4.09E-10	-2.54E-8	-7.90E-7
3	-1.37E-10	-2.87E-7	-1.86E-6	-2.78E-7	-6.13E-7	-1.90E-6	-1.07E-7	-4.24E-7	-1.87E-6	-3.61E-8	-3.40E-7	-1.86E-6

Table 2.148: The poles of $\bar{\bar{r}}_{q,\delta,\alpha,k}(t)$, (1.48), of (q, δ, α, k)-1-URA approximation, i.e. the rational approximation of $g(q, \delta, \alpha; t) = t^\alpha/(1 + q\, t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{4}$, and $q_0 = q_1 = 200$

k=4 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-4.37E-17	-3.58E-11	-2.27E-9	-1.82E-10	-2.74E-9	-7.28E-9	-3.67E-11	-6.13E-10	-3.29E-9	-7.15E-12	-1.70E-10	-2.49E-9
2	-2.74E-14	-3.92E-9	-2.70E-7	-1.94E-9	-2.61E-8	-3.85E-7	-4.95E-10	-1.29E-8	-3.03E-7	-1.21E-10	-7.39E-9	-2.78E-7
3	-3.60E-12	-5.10E-8	-1.01E-6	-2.87E-8	-1.49E-7	-1.03E-6	-9.62E-9	-9.51E-8	-1.02E-6	-2.96E-9	-6.97E-8	-1.02E-6
4	-7.23E-10	-5.50E-7	-2.12E-6	-7.20E-7	-1.21E-6	-2.20E-6	-3.69E-7	-8.72E-7	-2.15E-6	-1.65E-7	-6.92E-7	-2.13E-6

Table 2.149: The poles of $\bar{\bar{r}}_{q,\delta,\alpha,k}(t)$, (1.48), of (q,δ,α,k)-1-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{5}$, and $q_0 = q_1 = 200$

k=5 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-4.03E-18	-4.04E-12	-2.15E-10	-1.24E-10	-1.58E-9	-1.59E-9	-2.49E-11	-2.67E-10	-4.65E-10	-4.77E-12	-5.19E-11	-2.67E-10
2	-2.72E-15	-6.92E-10	-4.25E-8	-8.80E-10	-1.39E-8	-1.20E-7	-2.19E-10	-5.53E-9	-6.31E-8	-5.26E-11	-2.39E-9	-4.74E-8
3	-2.87E-13	-1.36E-8	-5.98E-7	-7.48E-9	-6.33E-8	-7.16E-7	-2.35E-9	-3.70E-8	-6.46E-7	-6.91E-10	-2.42E-8	-6.12E-7
4	-1.92E-11	-9.79E-8	-1.06E-6	-7.96E-8	-2.86E-7	-1.09E-6	-3.33E-8	-1.91E-7	-1.07E-6	-1.26E-8	-1.41E-7	-1.06E-6
5	-2.84E-9	-8.91E-7	-2.34E-6	-1.27E-6	-1.88E-6	-2.45E-6	-7.85E-7	-1.44E-6	-2.38E-6	-4.40E-7	-1.17E-6	-2.35E-6

Table 2.150: The poles of $\bar{\bar{r}}_{q,\delta,\alpha,k}(t)$, (1.48), of (q,δ,α,k)-1-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{6}$, and $q_0 = q_1 = 200$

k=6 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-4.32E-19	-4.96E-13	-2.40E-11	-9.16E-11	-9.85E-10	-4.79E-10	-1.82E-11	-1.34E-10	-9.38E-11	-3.45E-12	-1.98E-11	-3.78E-11
2	-3.19E-16	-1.07E-10	-4.83E-9	-4.91E-10	-8.19E-9	-3.07E-8	-1.20E-10	-2.58E-9	-1.08E-8	-2.80E-11	-8.37E-10	-6.26E-9
3	-3.37E-14	-3.43E-9	-2.03E-7	-2.95E-9	-3.36E-8	-4.16E-7	-8.78E-10	-1.76E-8	-2.86E-7	-2.49E-10	-9.76E-9	-2.28E-7
4	-1.62E-12	-2.89E-8	-7.60E-7	-2.00E-8	-1.17E-7	-7.85E-7	-7.50E-9	-7.37E-8	-7.77E-7	-2.62E-9	-5.10E-8	-7.67E-7
5	-7.59E-11	-1.60E-7	-1.11E-6	-1.60E-7	-4.57E-7	-1.16E-6	-7.97E-8	-3.21E-7	-1.13E-6	-3.63E-8	-2.42E-7	-1.11E-6
6	-8.98E-9	-1.27E-6	-2.51E-6	-1.87E-6	-2.51E-6	-2.62E-6	-1.28E-6	-2.02E-6	-2.56E-6	-8.24E-7	-1.69E-6	-2.52E-6

Table 2.151: The poles of $\bar{\bar{r}}_{q,\delta,\alpha,k}(t)$, (1.48), of (q,δ,α,k)-1-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{7}$, and $q_0 = q_1 = 200$

k=7 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	-3.97E-18	-6.61E-14	-3.04E-12	-7.13E-11	-6.54E-10	-1.85E-10	-1.41E-11	-7.54E-11	-2.58E-11	-2.63E-12	-9.04E-12	-7.08E-12
2	-2.69E-15	-1.58E-11	-5.92E-10	-3.12E-10	-5.15E-9	-8.76E-9	-7.44E-11	-1.30E-9	-2.20E-9	-1.70E-11	-3.21E-10	-9.69E-10
3	-2.86E-13	-7.27E-10	-3.45E-8	-1.46E-9	-2.01E-8	-1.81E-7	-4.19E-10	-9.13E-9	-8.03E-8	-1.15E-10	-4.15E-9	-4.76E-8
4	9.97E-25	-9.08E-9	-4.30E-7	-7.44E-9	-6.11E-8	-6.14E-7	-2.59E-9	-3.60E-8	-5.36E-7	-8.52E-10	-2.27E-8	-4.72E-7
5	-1.93E-11	-4.89E-8	-7.86E-7	-4.15E-8	-1.86E-7	-7.89E-7	-1.81E-8	-1.23E-7	-7.86E-7	-7.33E-9	-8.80E-8	-7.86E-7
6	-2.85E-9	-2.35E-7	-1.16E-6	-2.63E-7	-6.44E-7	-1.24E-6	-1.49E-7	-4.76E-7	-1.20E-6	-7.84E-8	-3.68E-7	-1.18E-6
7	9.43E-25	-1.65E-6	-2.63E-6	-2.50E-6	-3.07E-6	-2.75E-6	-1.80E-6	-2.56E-6	-2.69E-6	-1.26E-6	-2.20E-6	-2.65E-6

Table 2.152: The poles of $\bar{\bar{r}}_{q,\delta,\alpha,k}(t)$, (1.48), of (q,δ,α,k)-1-URA approximation, i.e. the rational approximation of $g(q,\delta,\alpha;t) = t^\alpha/(1+q\,t^\alpha)$ with functions from $\mathcal{R}_k$ on $[\delta, 1]$, $\alpha = 0.25, 0.5, 0.75$, $\mathbf{k} = \mathbf{8}$, and $q_0 = q_1 = 200$

k=8 / δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j / α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
1	1.52E-22	-9.55E-15	-4.29E-13	-5.76E-11	-4.55E-10	-8.51E-11	-1.13E-11	-4.61E-11	-9.06E-12	-2.09E-12	-4.72E-12	-1.74E-12
2	-1.70E-22	-2.40E-12	-8.16E-11	-2.17E-10	-3.40E-9	-3.00E-9	-5.05E-11	-7.02E-10	-5.68E-10	-1.13E-11	-1.37E-10	-1.89E-10
3	8.43E-27	-1.34E-10	-4.73E-9	-8.44E-10	-1.30E-8	-6.40E-8	-2.33E-10	-5.01E-9	-1.90E-8	-6.20E-11	-1.82E-9	-8.55E-9
4	-1.24E-27	-2.52E-9	-1.28E-7	-3.49E-9	-3.65E-8	-4.15E-7	-1.15E-9	-1.98E-8	-2.64E-7	-3.60E-10	-1.10E-8	-1.78E-7
5	2.05E-27	-1.73E-8	-5.85E-7	-1.53E-8	-9.62E-8	-6.38E-7	-6.08E-9	-6.04E-8	-6.34E-7	-2.28E-9	-4.05E-8	-6.13E-7
6	7.47E-27	-7.30E-8	-7.87E-7	-7.20E-8	-2.67E-7	-8.09E-7	-3.55E-8	-1.85E-7	-7.94E-7	-1.63E-8	-1.35E-7	-7.89E-7
7	3.06E-26	-3.19E-7	-1.22E-6	-3.81E-7	-8.31E-7	-1.30E-6	-2.37E-7	-6.40E-7	-1.26E-6	-1.39E-7	-5.09E-7	-1.24E-6
8	6.51E-24	-2.01E-6	-2.72E-6	-3.15E-6	-3.59E-6	-2.86E-6	-2.35E-6	-3.06E-6	-2.79E-6	-1.73E-6	-2.67E-6	-2.75E-6

2.8 Tables type (h) for 1-URA-decomposition coefficients

Table 2.153: The coefficients $\bar{\bar{c}}_j, j = 0, \ldots, k$ of the partial fraction representation (1.49) of (q, δ, α, k)-1-URA approximation (1.48) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{3}$, and $q_0 = q_1 = 100$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.88E-3	4.95E-3	4.98E-3	4.97E-3	4.97E-3	4.98E-3	4.97E-3	4.96E-3	4.98E-3	4.96E-3	4.95E-3	4.98E-3
1	-1.73E-14	-2.35E-9	-1.18E-7	-1.21E-9	-1.92E-8	-1.64E-7	-2.51E-10	-7.07E-9	-1.28E-7	-5.17E-11	-3.76E-9	-1.20E-7
2	-1.15E-11	-1.41E-7	-3.74E-6	-2.68E-8	-3.10E-7	-3.93E-6	-7.54E-9	-2.08E-7	-3.79E-6	-2.12E-9	-1.65E-7	-3.75E-6
3	-3.78E-9	-2.02E-6	-8.77E-6	-1.19E-6	-3.24E-6	-8.85E-6	-4.97E-7	-2.53E-6	-8.79E-6	-1.94E-7	-2.21E-6	-8.78E-6

Table 2.154: The coefficients $\bar{\bar{c}}_j, j = 0, \ldots, k$ of the partial fraction representation (1.49) of (q, δ, α, k)-1-URA approximation (1.48) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{4}$, and $q_0 = q_1 = 100$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.92E-3	4.97E-3	4.99E-3	4.98E-3	4.98E-3	4.99E-3	4.97E-3	4.97E-3	4.99E-3	4.97E-3	4.97E-3	4.99E-3
1	-1.23E-15	-2.31E-10	-8.64E-9	-7.02E-10	-7.79E-9	-1.94E-8	-1.42E-10	-1.88E-9	-1.09E-8	-2.77E-11	-6.48E-10	-9.11E-9
2	-7.73E-13	-2.67E-8	-1.14E-6	-8.06E-9	-1.19E-7	-1.49E-6	-2.16E-9	-6.32E-8	-1.23E-6	-5.73E-10	-4.03E-8	-1.16E-6
3	-9.92E-11	-3.57E-7	-4.87E-6	-1.21E-7	-7.57E-7	-4.94E-6	-4.35E-8	-5.37E-7	-4.89E-6	-1.49E-8	-4.31E-7	-4.88E-6
4	-1.80E-8	-3.55E-6	-9.58E-6	-3.00E-6	-5.82E-6	-9.74E-6	-1.62E-6	-4.65E-6	-9.62E-6	-8.05E-7	-4.02E-6	-9.59E-6

Table 2.155: The coefficients $\bar{\bar{c}}_j, j = 0, \ldots, k$ of the partial fraction representation (1.49) of (q, δ, α, k)-1-URA approximation (1.48) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{5}$, and $q_0 = q_1 = 100$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.94E-3	4.98E-3	4.99E-3	4.98E-3	4.98E-3	4.99E-3	4.98E-3	4.98E-3	4.99E-3	4.97E-3	4.98E-3	4.99E-3
1	-1.07E-16	-2.38E-11	-7.55E-10	-4.69E-10	-3.72E-9	-3.42E-9	-9.24E-11	-6.43E-10	-1.28E-9	-1.74E-11	-1.49E-10	-8.64E-10
2	-7.29E-14	-4.26E-9	-1.52E-7	-3.56E-9	-5.39E-8	-3.38E-7	-9.16E-10	-2.17E-8	-1.99E-7	-2.31E-10	-1.03E-8	-1.63E-7
3	-7.66E-12	-8.99E-8	-2.66E-6	-3.11E-8	-3.02E-7	-3.14E-6	-1.02E-8	-1.90E-7	-2.83E-6	-3.28E-9	-1.34E-7	-2.70E-6
4	-4.91E-10	-6.47E-7	-5.02E-6	-3.31E-7	-1.37E-6	-5.05E-6	-1.46E-7	-1.01E-6	-5.03E-6	-6.06E-8	-8.12E-7	-5.02E-6
5	-6.21E-8	-5.25E-6	-1.02E-5	-5.23E-6	-8.47E-6	-1.04E-5	-3.33E-6	-7.00E-6	-1.03E-5	-1.99E-6	-6.09E-6	-1.02E-5

Table 2.156: The coefficients $\bar{\bar{c}}_j, j = 0, \ldots, k$ of the partial fraction representation (1.49) of (q, δ, α, k)-1-URA approximation (1.48) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k} = \mathbf{6}$, and $q_0 = q_1 = 100$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.96E-3	4.98E-3	4.99E-3	4.98E-3	4.99E-3	4.99E-3	4.98E-3	4.98E-3	4.99E-3	4.98E-3	4.98E-3	4.99E-3
1	-1.07E-17	-2.67E-12	-7.85E-11	-3.40E-10	-2.01E-9	-8.77E-10	-6.57E-11	-2.71E-10	-2.15E-10	-1.20E-11	-4.57E-11	-1.07E-10
2	-8.01E-15	-5.91E-10	-1.57E-8	-1.95E-9	-2.70E-8	-6.76E-8	-4.83E-10	-8.16E-9	-2.80E-8	-1.16E-10	-2.83E-9	-1.86E-8
3	-8.51E-13	-2.06E-8	-7.47E-7	-1.20E-8	-1.48E-7	-1.45E-6	-3.73E-9	-7.90E-8	-9.88E-7	-1.13E-9	-4.59E-8	-8.13E-7
4	-4.02E-11	-1.85E-7	-3.54E-6	-8.25E-8	-5.51E-7	-3.72E-6	-3.23E-8	-3.74E-7	-3.64E-6	-1.22E-8	-2.77E-7	-3.57E-6
5	-1.75E-9	-9.93E-7	-5.10E-6	-6.58E-7	-2.10E-6	-5.18E-6	-3.41E-7	-1.60E-6	-5.12E-6	-1.66E-7	-1.30E-6	-5.10E-6
6	-1.67E-7	-6.93E-6	-1.07E-5	-7.62E-6	-1.09E-5	-1.10E-5	-5.31E-6	-9.26E-6	-1.08E-5	-3.58E-6	-8.17E-6	-1.07E-5

Table 2.157: The coefficients $\bar{\bar{c}}_j, j = 0, \ldots, k$ of the partial fraction representation (1.49) of (q, δ, α, k)-1-URA approximation (1.48) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 7}$, and $q_0 = q_1 = 100$

k=7 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	4.96E-3	4.98E-3	4.99E-3	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	4.99E-3
1	-1.19E-18	-3.29E-13	-9.39E-12	-2.61E-10	-1.19E-9	-3.00E-10	-4.95E-11	-1.34E-10	-5.06E-11	-8.77E-12	-1.77E-11	-1.72E-11
2	-9.61E-16	-8.00E-11	-1.81E-9	-1.22E-9	-1.46E-8	-1.65E-8	-2.92E-10	-3.39E-9	-4.95E-9	-6.74E-11	-8.79E-10	-2.57E-9
3	-1.10E-13	-3.90E-9	-1.07E-7	-5.90E-9	-8.05E-8	-4.55E-7	-1.73E-9	-3.51E-8	-2.08E-7	-4.97E-10	-1.61E-8	-1.35E-7
4	-4.90E-12	-5.41E-8	-1.72E-6	-3.04E-8	-2.77E-7	-2.61E-6	-1.10E-8	-1.71E-7	-2.14E-6	-3.84E-9	-1.12E-7	-1.87E-6
5	-1.47E-10	-3.03E-7	-3.74E-6	-1.70E-7	-8.53E-7	-3.70E-6	-7.68E-8	-6.06E-7	-3.73E-6	-3.31E-8	-4.62E-7	-3.74E-6
6	-4.96E-9	-1.38E-6	-5.21E-6	-1.07E-6	-2.85E-6	-5.35E-6	-6.26E-7	-2.24E-6	-5.27E-6	-3.47E-7	-1.85E-6	-5.23E-6
7	-3.66E-7	-8.48E-6	-1.11E-5	-1.01E-5	-1.31E-5	-1.14E-5	-7.41E-6	-1.13E-5	-1.12E-5	-5.36E-6	-1.01E-5	-1.11E-5

Table 2.158: The coefficients $\bar{\bar{c}}_j, j = 0, \ldots, k$ of the partial fraction representation (1.49) of (q, δ, α, k)-1-URA approximation (1.48) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 8}$, and $q_0 = q_1 = 100$

k=8 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	3.49E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3	4.98E-3	4.99E-3	5.00E-3
1	7.72E-20	-4.44E-14	-1.26E-12	-2.08E-10	-7.65E-10	-1.27E-10	-3.89E-11	-7.45E-11	-1.57E-11	-6.74E-12	-8.18E-12	-3.67E-12
2	1.80E-17	-1.12E-11	-2.38E-10	-8.36E-10	-8.45E-9	-5.05E-9	-1.94E-10	-1.56E-9	-1.14E-9	-4.31E-11	-3.16E-10	-4.46E-10
3	1.09E-20	-6.50E-10	-1.37E-8	-3.36E-9	-4.65E-8	-1.25E-7	-9.45E-10	-1.63E-8	-4.13E-8	-2.59E-10	-5.78E-9	-2.11E-8
4	-5.72E-27	-1.34E-8	-4.03E-7	-1.41E-8	-1.57E-7	-1.37E-6	-4.78E-9	-8.53E-8	-7.86E-7	-1.57E-9	-4.75E-8	-5.30E-7
5	1.50E-13	-1.02E-7	-2.55E-6	-6.23E-8	-4.34E-7	-3.02E-6	-2.56E-8	-2.88E-7	-2.87E-6	-1.01E-8	-2.02E-7	-2.69E-6
6	1.27E-25	-4.36E-7	-3.71E-6	-2.94E-7	-1.19E-6	-3.67E-6	-1.49E-7	-8.76E-7	-3.68E-6	-7.20E-8	-6.81E-7	-3.70E-6
7	3.58E-25	-1.78E-6	-5.34E-6	-1.55E-6	-3.59E-6	-5.52E-6	-9.82E-7	-2.91E-6	-5.42E-6	-5.99E-7	-2.44E-6	-5.37E-6
8	7.10E-24	-9.87E-6	-1.13E-5	-1.28E-5	-1.51E-5	-1.17E-5	-9.60E-6	-1.32E-5	-1.15E-5	-7.23E-6	-1.19E-5	-1.14E-5

Table 2.159: The coefficients $\bar{\bar{c}}_j, j = 0, \ldots, k$ of the partial fraction representation (1.49) of (q, δ, α, k)-1-URA approximation (1.48) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 3}$, and $q_0 = q_1 = 200$

k=3 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.45E-3	2.48E-3	2.49E-3	2.49E-3	2.49E-3	2.49E-3	2.49E-3	2.49E-3	2.49E-3	2.49E-3	2.48E-3	2.49E-3
1	-5.88E-16	-3.33E-10	-2.80E-8	-3.03E-10	-5.31E-9	-4.72E-8	-6.15E-11	-1.70E-9	-3.22E-8	-1.22E-11	-7.25E-10	-2.89E-8
2	-3.94E-13	-1.92E-8	-7.86E-7	-6.25E-9	-5.92E-8	-8.46E-7	-1.63E-9	-3.54E-8	-8.02E-7	-4.09E-10	-2.54E-8	-7.90E-7
3	-1.37E-10	-2.87E-7	-1.86E-6	-2.78E-7	-6.13E-7	-1.90E-6	-1.07E-7	-4.24E-7	-1.87E-6	-3.61E-8	-3.40E-7	-1.86E-6

Table 2.160: The coefficients $\bar{\bar{c}}_j, j = 0, \ldots, k$ of the partial fraction representation (1.49) of (q, δ, α, k)-1-URA approximation (1.48) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 4}$, and $q_0 = q_1 = 200$

k=4 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.47E-3	2.49E-3	2.50E-3	2.49E-3	2.49E-3	2.50E-3	2.49E-3	2.49E-3	2.50E-3	2.49E-3	2.49E-3	2.50E-3
1	-4.37E-17	-3.58E-11	-2.27E-9	-1.82E-10	-2.74E-9	-7.28E-9	-3.67E-11	-6.13E-10	-3.29E-9	-7.15E-12	-1.70E-10	-2.49E-9
2	-2.74E-14	-3.92E-9	-2.70E-7	-1.94E-9	-2.61E-8	-3.85E-7	-4.95E-10	-1.29E-8	-3.03E-7	-1.21E-10	-7.39E-9	-2.78E-7
3	-3.60E-12	-5.10E-8	-1.01E-6	-2.87E-8	-1.49E-7	-1.03E-6	-9.62E-9	-9.51E-8	-1.02E-6	-2.96E-9	-6.97E-8	-1.02E-6
4	-7.23E-10	-5.50E-7	-2.12E-6	-7.20E-7	-1.21E-6	-2.20E-6	-3.69E-7	-8.72E-7	-2.15E-6	-1.65E-7	-6.92E-7	-2.13E-6

Table 2.161: The coefficients $\bar{\bar{c}}_j, j = 0, \ldots, k$ of the partial fraction representation (1.49) of (q, δ, α, k)-1-URA approximation (1.48) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 5}$, and $q_0 = q_1 = 200$

k=5 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.48E-3	2.49E-3	2.50E-3	2.49E-3	2.50E-3	2.50E-3	2.49E-3	2.49E-3	2.50E-3	2.49E-3	2.49E-3	2.50E-3
1	-4.03E-18	-4.04E-12	-2.15E-10	-1.24E-10	-1.58E-9	-1.59E-9	-2.49E-11	-2.67E-10	-4.65E-10	-4.77E-12	-5.19E-11	-2.67E-10
2	-2.72E-15	-6.92E-10	-4.25E-8	-8.80E-10	-1.39E-8	-1.20E-7	-2.19E-10	-5.53E-9	-6.31E-8	-5.26E-11	-2.39E-9	-4.74E-8
3	-2.87E-13	-1.36E-8	-5.98E-7	-7.48E-9	-6.33E-8	-7.16E-7	-2.35E-9	-3.70E-8	-6.46E-7	-6.91E-10	-2.42E-8	-6.12E-7
4	-1.92E-11	-9.79E-8	-1.06E-6	-7.96E-8	-2.86E-7	-1.09E-6	-3.33E-8	-1.91E-7	-1.07E-6	-1.26E-8	-1.41E-7	-1.06E-6
5	-2.84E-9	-8.91E-7	-2.34E-6	-1.27E-6	-1.88E-6	-2.45E-6	-7.85E-7	-1.44E-6	-2.38E-6	-4.40E-7	-1.17E-6	-2.35E-6

Table 2.162: The coefficients $\bar{\bar{c}}_j, j = 0, \ldots, k$ of the partial fraction representation (1.49) of (q, δ, α, k)-1-URA approximation (1.48) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 6}$, and $q_0 = q_1 = 200$

k=6 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.48E-3	2.49E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.49E-3	2.50E-3	2.50E-3	2.49E-3	2.50E-3	2.50E-3
1	-4.32E-19	-4.96E-13	-2.40E-11	-9.16E-11	-9.85E-10	-4.79E-10	-1.82E-11	-1.34E-10	-9.38E-11	-3.45E-12	-1.98E-11	-3.78E-11
2	-3.19E-16	-1.07E-10	-4.83E-9	-4.91E-10	-8.19E-9	-3.07E-8	-1.20E-10	-2.58E-9	-1.08E-8	-2.80E-11	-8.37E-10	-6.26E-9
3	-3.37E-14	-3.43E-9	-2.03E-7	-2.95E-9	-3.36E-8	-4.16E-7	-8.78E-10	-1.76E-8	-2.86E-7	-2.49E-10	-9.76E-9	-2.28E-7
4	-1.62E-12	-2.89E-8	-7.60E-7	-2.00E-8	-1.17E-7	-7.85E-7	-7.50E-9	-7.37E-8	-7.77E-7	-2.62E-9	-5.10E-8	-7.67E-7
5	-7.59E-11	-1.60E-7	-1.11E-6	-1.60E-7	-4.57E-7	-1.16E-6	-7.97E-8	-3.21E-7	-1.13E-6	-3.63E-8	-2.42E-7	-1.11E-6
6	-8.98E-9	-1.27E-6	-2.51E-6	-1.87E-6	-2.51E-6	-2.62E-6	-1.28E-6	-2.02E-6	-2.56E-6	-8.24E-7	-1.69E-6	-2.52E-6

Table 2.163: The coefficients $\bar{\bar{c}}_j, j = 0, \ldots, k$ of the partial fraction representation (1.49) of (q, δ, α, k)-1-URA approximation (1.48) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 7}$, and $q_0 = q_1 = 200$

k=7 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	2.48E-3	2.49E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.49E-3	2.50E-3	2.50E-3
1	-3.97E-18	-6.61E-14	-3.04E-12	-7.13E-11	-6.54E-10	-1.85E-10	-1.41E-11	-7.54E-11	-2.58E-11	-2.63E-12	-9.04E-12	-7.08E-12
2	-2.69E-15	-1.58E-11	-5.92E-10	-3.12E-10	-5.15E-9	-8.76E-9	-7.44E-11	-1.30E-9	-2.20E-9	-1.70E-11	-3.21E-10	-9.69E-10
3	-2.86E-13	-7.27E-10	-3.45E-8	-1.46E-9	-2.01E-8	-1.81E-7	-4.19E-10	-9.13E-9	-8.03E-8	-1.15E-10	-4.15E-9	-4.76E-8
4	9.97E-25	-9.08E-9	-4.30E-7	-7.44E-9	-6.11E-8	-6.14E-7	-2.59E-9	-3.60E-8	-5.36E-7	-8.52E-10	-2.27E-8	-4.72E-7
5	-1.93E-11	-4.89E-8	-7.86E-7	-4.15E-8	-1.86E-7	-7.89E-7	-1.81E-8	-1.23E-7	-7.86E-7	-7.33E-9	-8.80E-8	-7.86E-7
6	-2.85E-9	-2.35E-7	-1.16E-6	-2.63E-7	-6.44E-7	-1.24E-6	-1.49E-7	-4.76E-7	-1.20E-6	-7.84E-8	-3.68E-7	-1.18E-6
7	9.43E-25	-1.65E-6	-2.63E-6	-2.50E-6	-3.07E-6	-2.75E-6	-1.80E-6	-2.56E-6	-2.69E-6	-1.26E-6	-2.20E-6	-2.65E-6

Table 2.164: The coefficients $\bar{\bar{c}}_j, j = 0, \ldots, k$ of the partial fraction representation (1.49) of (q, δ, α, k)-1-URA approximation (1.48) for $\delta = 0, 10^{-6}, 10^{-7}, 10^{-8}$, $\alpha = 0.25, 0.50, 0.75$, $\mathbf{k = 8}$, and $q_0 = q_1 = 200$

k=8 /δ	0.00	0.00	0.00	10^{-6}	10^{-6}	10^{-6}	10^{-7}	10^{-7}	10^{-7}	10^{-8}	10^{-8}	10^{-8}
j /α	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75	0.25	0.50	0.75
0	1.66E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3	2.50E-3
1	1.52E-22	-9.55E-15	-4.29E-13	-5.76E-11	-4.55E-10	-8.51E-11	-1.13E-11	-4.61E-11	-9.06E-12	-2.09E-12	-4.72E-12	-1.74E-12
2	-1.70E-22	-2.40E-12	-8.16E-11	-2.17E-10	-3.40E-9	-3.00E-9	-5.05E-11	-7.02E-10	-5.68E-10	-1.13E-11	-1.37E-10	-1.89E-10
3	8.43E-27	-1.34E-10	-4.73E-9	-8.44E-10	-1.30E-8	-6.40E-8	-2.33E-10	-5.01E-9	-1.90E-8	-6.20E-11	-1.82E-9	-8.55E-9
4	-1.24E-27	-2.52E-9	-1.28E-7	-3.49E-9	-3.65E-8	-4.15E-7	-1.15E-9	-1.98E-8	-2.64E-7	-3.60E-10	-1.10E-8	-1.78E-7
5	2.05E-27	-1.73E-8	-5.85E-7	-1.53E-8	-9.62E-8	-6.38E-7	-6.08E-9	-6.04E-8	-6.34E-7	-2.28E-9	-4.05E-8	-6.13E-7
6	7.47E-27	-7.30E-8	-7.87E-7	-7.20E-8	-2.67E-7	-8.09E-7	-3.55E-8	-1.85E-7	-7.94E-7	-1.63E-8	-1.35E-7	-7.89E-7
7	3.06E-26	-3.19E-7	-1.22E-6	-3.81E-7	-8.31E-7	-1.30E-6	-2.37E-7	-6.40E-7	-1.26E-6	-1.39E-7	-5.09E-7	-1.24E-6
8	6.51E-24	-2.01E-6	-2.72E-6	-3.15E-6	-3.59E-6	-2.86E-6	-2.35E-6	-3.06E-6	-2.79E-6	-1.73E-6	-2.67E-6	-2.75E-6

2.9 How to access the BURA data from the repository

2.9.1 Data generated by Remez algorithm

Remez algorithm is implemented according to the results reported in [41]. It is part of a software developed in IICT, Bulgarian Academy of Sciences, concerting the more general problem of the best uniform rational approximation of a given function of $x \in [0,1]$ by $P_m(x)/Q_k(x)$, where P_m and Q_k are polynomials of degree $m \geq k$, correspondingly. In this handbook we use only the particular case when $m = k$. To make the writing more concise we use the following notations

$$P_n(x) \quad \text{denoted by} \quad p(n, F; x) = \sum_{j=0}^{n} F(j) * x^j,$$

where n is the degree of the polynomial and $F(j)$, $j = 0, \dots, n$, are the coefficients. Further, we denote the zeros of $Q_k(x)$ by $U0(j))$, $j = 0, \dots, k$, which in general are complex numbers

$$U0(j)) = Re\{U0(j))\} + iIm\{U0(j))\}, \quad j = 1, \dots, k, \quad i^2 = -1.$$

Accordingly, the best uniform rational approximation $P_m(x)/Q_k(x) = p(m, A; x)/p(k, B; x)$ in the data files will be denoted by and represented as a sum of partial fractions as

$$\begin{aligned} p(m, A; x)/p(k, B; x) &= p(m-k, C; x) + p(k-1, D; x)/p(k, B; x) \\ &= \sum_{j=0}^{m-k} C(j) * x^j + \sum_{j=1}^{k} E(j)/(x - U0(j)). \end{aligned} \tag{2.1}$$

where the coefficients $C(j)$ and $E(j)$ are in general complex numbers. Of course, when the poles are real, then the complex parts, since obtained by computational procedure for finding roots of a polynomial, will have very small imaginary parts.

In our case $k = m$ and the above representation becomes

$$r(m, k, A, B, x) = C(0) + \sum_{j=1}^{k} E(j)/(x - U0(j)).$$

2.9.2 Examples how to use the data in the repository

Now we shall illustrate on a couple of examples how to solve the system $\widetilde{\mathbb{A}}^\alpha \widetilde{u}_h = \widetilde{f}_h$ arising in the approximation of the Example 1 that produces the matrix $\widetilde{\mathbb{A}}$ defined by (1.10) by using BURA method (1.39). For definiteness and convenience, we assume that $h = 10^{-3}$, $\min a(x) = 0.25/\pi^2$ and $\max a(x) = 1$, so that $\lambda_1 \geq 1$ and $\lambda_1/\lambda_N = h^2/4 = 0.25 * 10^{-6}$. We assume also that the data f is such that the step-size $h = 10^{-3}$ allows the following estimate for the semi-discrete error $\|u - u_h\| \leq 10^{-3}\|f\|$.

The task we have is to find an approximation $\widetilde{w}_h$ of the solution $\widetilde{u}_h$ with relative error 10^{-3} by using (1.39) so that (simplified due to $\lambda_1 = 1$)

$$\widetilde{w}_h = r_{\alpha,k}(\widetilde{\mathbb{A}}^{-1})\widetilde{f}_h = \left(c_0 + \sum_{i=1}^{k} c_i(\widetilde{\mathbb{A}}^{-1} - d_i)^{-1}\right)\widetilde{f}_h = \left(\widetilde{c}_0 + \sum_{i=1}^{k} \widetilde{c}_i(\widetilde{\mathbb{A}} - \widetilde{d}_i)^{-1}\right)\widetilde{f}_h.$$

The coefficients c_j, $j = 0, \dots, k$, and the poles d_j, $k = 1, \dots, k$ will be obtained from the repository `http://parallel.bas.bg/lcst/BURA/`. Moreover, in the repository one can find more information about $r_{\alpha,k}(t)$, namely, the extremal points of the error $t^\alpha - r_{\alpha,k}(t)$, the zeros of the numerator, etc.

Example 1. We need to solve $\widetilde{\mathbb{A}}^{\alpha}\widetilde{u}_h = \widetilde{f}_h$ with relative accuracy 10^{-3} for $\delta = 0$ and $\alpha = 0.25$. We visit Table 2.2 and look for the corresponding error for $q = 0$. It appears that error below 10^{-3} is achieved for $k = 7$. In order to implement our algorithm we need to get the poles d_j from the second row of Table 2.37 (under $\delta = 0$ and $\alpha = 0.25$), and the coefficients c_j from the second row of Table 2.67 (under $\delta = 0$ and $\alpha = 0.25$) of BURA. According to Table 2.1, the corresponding tables are encoded as $Cq000d0a25k7p$ and $Dq000d0a25k7p$. To recover the data, in a browser open the site of the repository `http://parallel.bas.bg/lcst/BURA/` which will get you to the html-repository.

```
Index
mode   size  last-canged   name
dr-x         Jul 18 09:26  BURA-tabl/ - with extremal points
dr-x         Jul 18 10:50  BURA-dcmp/ - with poles an decomposition coefficients for BURA
dr-x         Jul 18 11:22  0URA-dcmp/ - with poles an decomposition coefficients for 0URA
dr-x         Jul 18 11:35  1URA-dcmp/ - with poles an decomposition coefficients for 1URA
-r--   672k  Sep 21 11:51  BURA-data-report.pdf - 60 pages document
-r--   3.2M  Sep 21 11:51  all-in-one.zip  - archive with all files (approx 2000)
-r--   165k  Sep 21 11:51  all-in-one_long.lst  -  long  list of all folders and sub-folders
-r--    720  Sep 21 11:51  all-in-one_short.lst -  short list of the main folder
```

Figure 2.1: Index of the file repository

The coefficients in the partial fraction representation of BURA are in the folder `BURA-dcmp/`. Consequently, from `http://parallel.bas.bg/lcst/BURA/BURA-dcmp/` you get a long list of data files where you can extract the data from.

```
Index
mode   size  last-canged   name
dr-x         Aug  7 12:19  ../
dr-x         Aug  7 13:16  add/
-r--    229  Jul 18 07:26  0-data-info.txt
-r--    20k  Aug 18 07:26  0-data-info.pdf
-r--    901  Jul 18 09:28  q000d0a25k3.tab
-r--   1.1k  Jul 18 09:28  q000d0a25k4.tab
-r--   1.3k  Jul 18 09:28  q000d0a25k5.tab
-r--   1.5k  Jul 18 09:28  q000d0a25k6.tab
-r--   1.7k  Jul 18 09:28  q000d0a25k7.tab
-r--   1.9k  Jul 18 09:28  q000d0a25k8.tab
. . .
```

Figure 2.2: The list of file containing the BURA data

Note that the name of the file corresponds for the data you need. In order to understand better how to recover the needed coefficient we recommend you read to file `0-data-info.pdf`. Now, if you need BURA for $q = 0$, $\delta = 0$, $\alpha = 0.25$, and $k = 7$ you open the file `q000d0a25k7.tab` to get the data shown on Figure 2.3.

Comparing this explained by formula (2.1) with the representation of $r_{\alpha,k}(x)$ by (1.35) we realize that the poles d_j are in $Re\{U0(j)\}$, while the coefficients c_j are in $Re\{E(j)\}$, $j = 1, \ldots, 7$ except $c_0 = C(0)$.

Note that the poles of the rational function $r_{\alpha,k}(t)$ are computed by finding the roots of its denominator. In the general case these are complex numbers with real and complex parts. However, in our case the roots are real, so the fact that the complex parts are near zero is another indication that our computations are correct. The same remark is valid also for the coefficients in the partial fraction representation. One can recover the data including the complex parts of the roots and coefficients from the file with the same name abut extension `.txt`. Thus, instead of file `q000d0a25k7.tab` we need to access file `q000d0a25k7.txt` The full path is `http://parallel.bas.bg/lcst/BURA/BURA-dcmp/add/`. According Remark 1.5.1 and (1.38) we obtain for the coefficients $\{\widetilde{c}_j\}_{j=0}^{k}$ and for the poles $\{\widetilde{d}_j\}_{j=1}^{k}$, shown on Figure 2.5.

```
........................
    0 C(j), j=0,M-K
    0,  1.5251198659461835471669134E+0000,
.......................
    7 Re{U0(j)}, j=1,K
    1, -2.2376872996078341567977828E-0010,
    2, -7.8486161880478847863452285E-0008,
    3, -5.5731465614475310569044313E-0006,
    4, -1.8879914699444446592534226E-0004,
    5, -4.0807806486154275568196059E-0003,
    6, -6.6631003383177437492683783E-0002,
    7, -1.3009123561707709448297558E+0000,
    7 Re{E(j)}, j=1,K
    1, -1.4685691492826336539266828E-0012,
    2, -1.4250266060744046359349294E-0009,
    3, -2.3302001273674185907801704E-0007,
    4, -1.6270756180817065256760833E-0005,
    5, -6.7439995512918544429211685E-0004,
    6, -2.0780142734679021643365256E-0002,
    7, -1.1636545994360130292202863E+0000.
```

Figure 2.3: BURA Data for Example 1

$$
\begin{aligned}
c_0 &= \quad C(0) \quad = \quad 1.5251198659461835\\
c_1 &= Re\{E(1)\} = -1.468569149282634 * 10^{-12},\\
c_2 &= Re\{E(2)\} = -1.425026606074404 * 10^{-9},\\
c_3 &= Re\{E(3)\} = -2.330200127367418 * 10^{-7},\\
c_4 &= Re\{E(4)\} = -1.627075618081706 * 10^{-5},\\
c_5 &= Re\{E(5)\} = -6.743999551291854 * 10^{-4},\\
c_6 &= Re\{E(6)\} = -2.078014273467902 * 10^{-2},\\
c_7 &= Re\{E(7)\} = -1.1636545994360130,
\end{aligned}
$$

(a) Coefficients c_j with 15 significant digits

(recall $d_j = Re\{U0(j)\}$, $j = 1, \ldots, 7$)

$$
\begin{aligned}
d_1 &= Re\{U0(1)\} = -2.237687299607834 * 10^{-10},\\
d_2 &= Re\{U0(2)\} = -7.848616188047885 * 10^{-8},\\
d_3 &= Re\{U0(3)\} = -5.573146561447531 * 10^{-6},\\
d_4 &= Re\{U0(4)\} = -1.887991469944444 * 10^{-4},\\
d_5 &= Re\{U0(5)\} = -4.080780648615427 * 10^{-3},\\
d_6 &= Re\{U0(6)\} = -6.663100338317743 * 10^{-2},\\
d_7 &= Re\{U0(7)\} = -1.300912356170771.
\end{aligned}
$$

(b) The poles d_j with 15 significant digits

Figure 2.4: The coefficients and poles of BURA for Example 1

```
  j,  coeffs  \tilde c_j, j=0,...,k           j,    poles \tilde d_j, j=1,...,k
--------------------------------------      ---------------------------------------
  0,  7.8649908986141400766246111E-0004,
  1,  6.8758756374255199468223446E-0001,      1, -7.6869129212016636558276595E-0001,
  2,  4.6805384769874068750186188E+0000,      2, -1.5008028533643146339273005E+0001,
  3,  4.0497762636194054740290213E+0001,      3, -2.4505115224443418015306629E+0002,
  4,  4.5646520861012335445240629E+0002,      4, -5.2966340998851315319920697E+0003,
  5,  7.5022631473305133482030654E+0003,      5, -1.7943185038727329359280164E+0005,
  6,  2.3133257355683798168384452E+0005,      6, -1.2741099526854566986420878E+0007,
  7,  2.9328888647334126024639520E+0007,      7, -4.4688996544568804592027580E+0009.
```

Figure 2.5: The coefficients $\widetilde{c}_j$ (left) and the poles $\widetilde{d}_j$ (right) of BURA in $[1, \infty)$ for Example 1; recall $k = 7$

Example 2 We want to solve $\widetilde{\mathbb{A}}^{\alpha}\widetilde{u}_h = \widetilde{f}_h$ with relative accuracy 10^{-4} for $\alpha = 0.50$. Since $\lambda_N/\lambda_1 = 2.5 * 10^{-7} > 10^{-7}$. Thus, we can choose any $\delta \leq 10^{-7}$. We shall choose $\delta = 10^{-8}$, which indeed is smaller than λ_N/λ_1. Visiting Table 2.2 and looking for the corresponding error for $q = 0$ and $\delta = 10^{-8}$ we see that $k = 6$ gives the desired accuracy. Then again, to implement the algorithm through the form (1.35) we need to get the poles $d_j, j = 1, \ldots, 6$ from Table 2.36, row 6 (under $\delta = 10^{-8}$ and $\alpha = 0.50$), and the coefficients $c_j, j = 0, 1, \ldots, 6$, from Table 2.66, row 6 (under $\delta = 10^{-8}$ and $\alpha = 0.50$). In order to recover them from the html-repository with 25 significant digits we read the file `q000d8a50k6.tab`:

```
..........................................
 0 C(j), j=0,M-K
    0,  3.2884110965902564982570418E+0000,
...........................................
    6 Re{U0(j)}, j=1,K
    1, -2.9026635633198611614392944E-0006,
    2, -1.2844649138280910668994002E-0004,
    3, -2.2714606179594160697493911E-0003,
    4, -2.5644587248425397513332244E-0002,
    5, -2.3875648347625806119814403E-0001,
    6, -4.0426255143071180974523916E+0000,
    6 Re{E(j)}, j=1,K
    1, -7.3047263730915787436076796E-0009,
    2, -1.4856610579547499079286817E-0006,
    3, -8.9729276335619099229017533E-0005,
    4, -2.9805865012026772176372313E-0003,
    5, -8.4097511581798008760373107E-0002,
    6, -1.1182633938371627571773423E+0001.
```

Figure 2.6: The BURA Data for Example 2: recall $m = k = 6$

One can recover the data including the complex parts of the roots and coefficients from the file with the same name abut extension `.txt`. Thus, instead of file `q000d8a50k6.tab` we need to access file `q000d8a50k6.txt` to get (recall $m = k = 6$). As we see on Figure 2.8, the complex parts of the poles $U0(j)$ and of the coefficients $E(j)$ are almost zero. Therefore, we ignore them in table `q000d8a50k6.tab` as seen on Figure 2.6.

$c_0 = \quad C(0) \quad = \quad 3.288411096590256$
$c_1 = \quad Re\{E(1)\} = -7.304726373091578 * 10^{-9}$
$c_2 = \quad Re\{E(2)\} = -1.485661057954749 * 10^{-6}$
$c_3 = \quad Re\{E(3)\} = -8.972927633561909 * 10^{-5}$
$c_4 = \quad Re\{E(4)\} = -2.980586501202677 * 10^{-3}$
$c_5 = \quad Re\{E(5)\} = -8.409751158179800 * 10^{-2}$
$c_6 = \quad Re\{E(6)\} = -1.118263393837162 * 10$

(a) Coefficients c_j with 15 significant digits

(recall $d_j = Re\{U0(j)\}$, $j = 1, \ldots, 7$)
$d_1 = \quad Re\{U0(1)\} = -2.902663563319861 * 10^{-6}$
$d_2 = \quad Re\{U0(2)\} = -1.284464913828091 * 10^{-4}$
$d_3 = \quad Re\{U0(3)\} = -2.271460617959416 * 10^{-3}$
$d_4 = \quad Re\{U0(4)\} = -2.564458724842539 * 10^{-2}$
$d_5 = \quad Re\{U0(5)\} = -2.387564834762580 * 10^{-1}$
$d_6 = \quad Re\{U0(6)\} = -4.042625514307118 * 10$

(b) The poles d_j with 15 significant digits

Figure 2.7: The coefficients and the poles for Example 2

According Remark 1.5.1 and (1.38) we obtain the coefficients $\{\widetilde{c}_j\}_{j=0}^{k}$ and the poles $\{\widetilde{d}_j\}_{j=1}^{k}$, shown on Figure 2.9.

```
........................
0 C(j), j=0,M-K
    0,  3.2884110965902564982570418E+0000,
................................
    6 U0(j), j=1,K
    1, -2.9026635633198611614392944E-0006, -1.7520806294041483876573147E-0037
    2, -1.2844649138280910668994002E-0004,  1.8242239649606423471743024E-0037
    3, -2.2714606179594160697493911E-0003, -7.2143334828724323886476075E-0039
    4, -2.5644587248425397513332244E-0002, -7.2776229715659618428404602E-0047
    5, -2.3875648347625806119814403E-0001, -7.3384739154186567125843208E-0052
    6, -4.0426255143071180974523916E+0000, -8.8654988085625587825735093E-0067
    6 E(j), j=1,K
    1, -7.3047263730915787436076796E-0009,  5.7573111695991243362896879E-0037
    2, -1.4856610579547499079286817E-0006, -5.9992385756159082541115791E-0037
    3, -8.9729276335619099229017533E-0005,  2.4155906802799248832468797E-0038
    4, -2.9805865012026772176372313E-0003,  2.2938808341026098990357342E-0041
    5, -8.4097511581798008760373107E-0002,  9.4285923737400470688723456E-0042
    6, -1.1182633938371627571773423E+0001,  4.4663981643768036611005825E-0042
```

Figure 2.8: The BURA Data for Example 2 in complex numbers; recall $m = k = 6$

```
 j,  coeffs  \tilde c_j, j=0,...,k    j,
----------------------------------------
 0,  1.8619525184162842197896828E-0004,
 1,  6.8425358785323701384064700E-0001,
 2,  1.4752743712419678037242149E+0000,
 3,  4.5322129077686442621147161E+0000,
 4,  1.7390967446300787239267527E+0001,
 5,  9.0048244054276093870632027E+0001,
 6,  8.6698293594856505069077514E+0002.
```

(a) The coefficients $\widetilde{c}_j$

```
 j,    poles \tilde d_j, j=1,...,k
----------------------------------------
 1, -2.4736399561644631729004006E-0001,
 2, -4.1883679364017772326828007E+0000,
 3, -3.8994583547504784590380748E+0001,
 4, -4.4024536110969760472793286E+0002,
 5, -7.7853430579096145080903339E+0003,
 6, -3.4451116300101647683572511E+0005,
```

(b) The poles $\widetilde{d}_j$

Figure 2.9: Coefficients $\widetilde{c}_j$ and poles $\widetilde{d}_j$ of BURA in $[1, \infty)$ for Example 2, $k = 6$.

Conclusion

The information, provided in this handbook uses sophisticated analytical tools and advanced level of computations based on the best uniform rational approximations of t^α and $\frac{t^\alpha}{1+qt^\alpha}$, $t \in [\delta, 1]$, $0 \le \delta < 1$. Remez algorithm is used for the derivation of the BURA elements. Thus, the use of the method is NOT straightforward and requires some preliminary work to get all the necessary data, required by the proposed BURA methods.

The data, collected in the tables is very useful for saving the a priori off-line computational work, needed to minimize the number of solves of systems of the type $(\widetilde{\mathbb{A}} + \widetilde{c}_j \mathbb{I})\widetilde{w}_h = \widetilde{f}_h$. For example, such situation arises when one consecutively solves the semi-discrete problem $\widetilde{\mathbb{A}}^\alpha \widetilde{u}_h = \widetilde{f}_h$ with different right-hand sides.

Acknowledgments

This handbook has been partly support by the Grant No BG05M2OP001-1.001-0003, financed by the Science and Education for Smart Growth Operational Program (2014-2020) and co-financed by the EU through the European structural and Investment funds.

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The handbook is intended to be used as a complementary part of the courses on

"Numerical Methods for Sparse Linear Systems" and "Convex Analysis and its Applications to Image Processing" that are currently taught within the Master's program "Computational Mathematics and Mathematical Modelling" in Sofia University "St. Kliment Ohridski". The presented material can also serve as a short course on numerical methods for solving fractional diffusion problems and as an extension of existing master or PhD programs in the field of advanced scientific computations. The handbook considers the mathematical problem of solving systems equations involving fractional powers of self-adjoint elliptic operators. Due to the mathematical modelling of various non-local phenomena using such operators, a number of numerical solution methods have been introduced, studied, and tested in the last two decades. The handbook deals with the discrete counterpart of such problems obtained from finite difference or finite element approximations of the corresponding elliptic problems. All necessary information regarding the recently introduced by the authors' method based on best uniform rational approximation (BURA) of a proper scalar function in the unit interval is provided. A substantial part of the handbook is the presented 160 tables containing the related coefficients, zeros, poles, extreme points of the error function, the terms of the BURA partial fraction decomposition, etc. Links to website where one can download the files with the data characterizing the particular rational approximations, which are available with enough significant digits, are also provided so one can use them in his/her own computations. The examples are presented in a form ready for computer exercises.

eISSN 2367-8666